Mechanical Systems and Signal Processing (1998) **12**(2), 333–355

MINIMISATION OF THE EFFECT OF UNCERTAINTY ON MODEL ESTIMATION

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(Received October 1996, accepted after revisions July 1997)

The objective of this paper is to minimise the effect of an uncertain oil film bearing model on the estimation of a foundation model from run-down measurements on a turbo-generator. To do this a method is presented to identify a dynamic transfer matrix relating the response vector of the foundation at the bearings to an estimated force which is insensitive with respect to the oil film model. Analytically this transfer matrix is related to that of the foundation simply by the multiplication by a transformation matrix which only depends on the model of the rotor and on the model of the oil film, and which of course retains the sensitivity of the original problem with respect to the oil film model. But since the estimation error of the robust transfer matrix estimate is minimum the retained error is mainly due to the erroneous oil film model used in the transformation matrix. The use of the robust estimate to calculate the contribution of the support on the dynamics of the mounted machine for variations of the oil film model increases the reliability of the predictions.

1. INTRODUCTION

This paper reflects a partial result of research to develop a method to estimate the influence of elastic support on the dynamics of a mounted rotor. The elastic support consists of the foundation and the oil film of the journal bearings. The foundation here is defined as that dynamic system which remains when the rotor and the oil film are removed from the entire system. This remaining system includes the housings of the journal bearings as well as the pedestals, and a large amount of flexible connected parts, e.g. tubes, ladders, hand supports, etc. A central aspect in estimating the influence of the support on the dynamics of the rotor is the fact that the complex, multidimensional systems of the rotor and the foundation are only connected by the relatively simple low-dimensional system of the oil film of the journal bearings.

The rotor as well as the foundation can be divided into internal degrees of freedom (dof) and interface (connection) dof (Fig. 1). The model of the bearings is to be understood as the model of the oil film of the journal bearings and has no internal dof.

The influence of the support on the dynamics of the rotor at frequency $\omega \in \Omega$ is given by the inner force vector $f_{FB}(\omega)$ (see Fig. 1) of the foundation at the bearing locations. This force is in general not available through measurement, but it is possible to find an analytical expression for f_{FB} depending on the models of the rotor, bearings and foundation, and the service load of the machine, and the unbalance force on the rotor.

Spatially discretised models in the frequency domain are already established for the rotor as well as for the bearings, although that of the bearings has to be considered as uncertain. However, despite intensive efforts it has not yet been possible to generate a model for the foundation which reflects the dynamic behaviour with sufficient accuracy.

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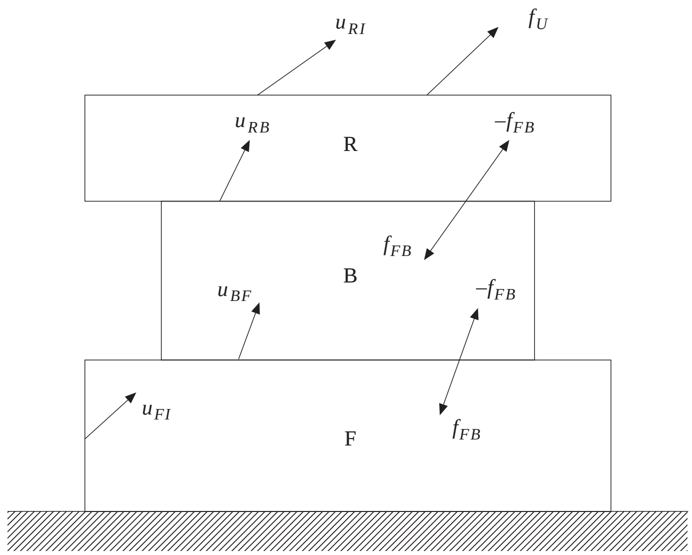

Figure 1. Schematic representation of the system: rotor (R), bearings (B) and foundation (F), and the corresponding responses and forces.

To estimate the unknown foundation model, the expression of the force f_{FB} is rewritten in the terms of the response vector $u_{FB}(\omega)$ of the foundation at the bearing locations. In order to establish an input/output equation to estimate the transfer matrix $F(\omega)$ of the foundation, the response u_{FB} is substituted by measurements u_{FB}^M to give an estimate f_{FB}^M of the force f_{FB}. This problem of model estimation has been already discussed in several papers [1–5], where the use of the estimate f_{FB}^M transfers mismatches between the bearing dynamics involved in the measured response u_{FB}^M and the initial bearing model to errors of foundation model estimate. Using this estimate of the foundation model within the expression of the force f_{FB} to study the contribution of the elastic support on the dynamics of the rotor under variation of the uncertain bearing model will lead to erroneous results.

The basic idea presented in this paper is to minimise the effect of the uncertainty of the initial bearing model on the model estimation process. The method presented here is based on the following two facts.

(i) To calculate the response u_{FB}, only a simulation model of the foundation has to be known. Thus, it is sufficient to model the foundation by a frequency filter rather than in terms of mass, damping and stiffness matrices.
(ii) The entire influence of the foundation on the dynamics of the rotor is given by the foundation force f_{FB} (see Fig. 1) which in general cannot be measured, but can be calculated depending on the response u_{FB} or on the foundation model, and on the model of the bearings and the rotor.

To minimise the effect of the uncertainty on the estimation process it is proposed to estimate a transfer matrix $H(\omega)$ which maps u_{FB} to a force f_H rather than to the force f_{FB}, which is sensitive with respect to the initial bearing model. Amongst all possible forces, f_H can be chosen to be of minimum sensitivity with respect to the initial bearing model. Analytically, the transfer matrix H is related to that of the foundation F by a transformation P via

$$H(\omega) = P(\omega)F(\omega). \tag{1}$$

$P(\omega)$ depends only on the models of the rotor and of the bearings. Since the estimate $\hat{F}$ of F is sensitive but the estimate $\hat{H}$ of H is insensitive with respect to the initial model of the bearings, the sensitivity is now mainly concentrated in the matrix P due to its dependency on the bearing model. Of course the transfer matrix H will depend non-linearly on the frequency ω; but since a filter model is sufficient, the estimation of H still leads to a linear inverse problem. Having estimated H, the response u_{FB} and the force f_{FB} can be calculated in terms of the estimate $\hat{H}$ and, of course, depends on the models of the rotor and of the bearing. The important point here is that in the expressions of the response and of the force, only $\hat{H}$ and P occur explicitly rather than $\hat{F}$. Of course, in the initial bearing model, the calculated response and force using the expression with the estimate $\hat{F}$ will not differ significantly from the response and force calculated by using the expression with the estimate $\hat{H}$. But in general, these expressions are different functions of the bearing model due to the occurence of P in the expressions involving $\hat{H}$. Thus, the results under variations of the bearing model will be mostly different. Since the estimate of F depends sensitively on the initial bearing model whilst the estimate of H is of minimum sensitivity, the error in the predicted responses and forces under variations of the bearing model will be smaller using the expressions involving $\hat{H}$ than those resulting from the formula involving $\hat{F}$. Due to the minimisation of the effect of the force sensitivity with respect to the initial bearing model on the problem of model estimation, the estimate of H provides more reliable predictions of the influence of the elastic support on the dynamics of the rotor than the estimate of F.

In Section 2, the inner force f_{FB} exerted at the bearings and the input/output equation to estimate the condensed dynamic stiffness matrix of the foundation are inferred from the entire system model. It is explained of how uncertainities in the model of the bearings are transferred to uncertainties in the foundation model estimate, and how this erroneous estimate influences the calculation of the force f_{FB}. The advantage of the use of the robust estimate $\hat{H}$ is explained. A method to find P is presented in Section 3. It starts with a generalisation of the condensation process to calculate the force f_{FB}. A general force expression is inferred which depends on the particular choice of the master and slave dof of the condensation process. The redundancies of this general force are then used to determine that force f_H which is of minimum sensitivity with respect to the initial bearing model. Two criteria are formulated. In Section 4, details of the method are demonstrated by a simple example of two elastomechanical models coupled via two massless springs. The mathematical description of the model of the entire system and of the data available for the estimation process are introduced in Appendix A. Appendix B contains the sensitivity analysis, and in Appendix C the frequency filter model and the estimation method of the filter parameters are explained.

2. PROBLEM AND SOLUTION

In order to estimate the transfer matrix F of the foundation in the input/output equation

$$F(\omega)u_{FB}(\omega) = f_{FB}(\omega), \quad \omega \in \Omega, \tag{2}$$

the response u_{FB} of the foundation at the bearing location is substituted by measurements u_{FB}^M. The transfer matrix F results from dynamic condensation to eliminate the unknown displacements u_{FI} of the internal foundation dof [see equation (A3)], which leads to

$$u_{FI} = -A_{FII}^{-1}A_{FIB}u_{FB} \tag{3}$$

and

$$f_{FB} = \underbrace{(A_{FBB} - A_{FBI}A_{FII}^{-1}A_{FIB})}_{=:\mathrm{F}}u_{FB}. \tag{4}$$

Coupling the models of the rotor from equation (A1) and the model of the bearings from equation (A2) leads the input/output equation of the synthesised model of the rotor mounted on the bearings

$$\begin{bmatrix} A_{RII} & A_{RIB} & 0 \\ A_{RBI} & A_{RBB}+B & -B \\ 0 & -B & B \end{bmatrix}\begin{pmatrix} u_{RI} \\ u_{RB} \\ u_{FB} \end{pmatrix} = \begin{pmatrix} f_{RI} \\ 0 \\ -f_{FB} \end{pmatrix} \in \mathbb{C}^{n+8m}, \tag{5}$$

which of course is coupled by u_{FB} and f_{FB} to equation (4). Equations (3), (4) and (5) are equivalent to the input/output equation (A6) of the entire model.

Equation (5) is then used to find an expression for f_{FB} in terms of the response u_{FB} and in terms of the models of the rotor and the bearings. Dynamic condensation is used to eliminate the unknown displacements u_{RI}, u_{RB} of the rotor. Equation (5) is equivalent to

$$Bu_{FB} - Bu_{RB} = -f_{FB}, \tag{6}$$

$$\begin{bmatrix} 0 \\ -B \end{bmatrix} u_{FB} + A_{RB}\begin{pmatrix} u_{RI} \\ u_{RB} \end{pmatrix} = \begin{pmatrix} f_{RI} \\ 0 \end{pmatrix}, \tag{7}$$

where the matrix A_{RB} is defined as

$$A_{RB} := \begin{bmatrix} A_{RII} & A_{RIB} \\ A_{RBI} & A_{RBB}+B \end{bmatrix}. \tag{8}$$

The dynamic stiffness matrix B of the bearings in the second diagonal block ensures that A_{RB} defined in equation (8) is non-singular, and equation (7) gives

$$\begin{pmatrix} u_{RI} \\ u_{RB} \end{pmatrix} = A_{RB}^{-1}\begin{pmatrix} f_{RI} \\ Bu_{FB} \end{pmatrix}. \tag{9}$$

Defining G as the inverse of A_{RB} and using the same partition, i.e.

$$\begin{bmatrix} A_{RII} & A_{RIB} \\ A_{RBI} & A_{RBB}+B \end{bmatrix}^{-1} = \begin{bmatrix} G_{II} & G_{IB} \\ G_{BI} & G_{BB} \end{bmatrix}, \tag{10}$$

an expression for u_{RB} then follows from equation (9),

$$u_{RB} = G_{BI}f_{RI} + G_{BB}Bu_{FB}, \tag{11}$$

that may be used in equation (6) to give

$$f_{FB} = \underbrace{[BG_{BB}B - B]}_{=:C}u_{FB} + BG_{BI}f_{RI}. \tag{12}$$

This is an expression for the force acting on the foundation at the bearings. It depends on the model of the bearings B, on the model of the rotor, both contained in G_{BB} and in

G_{BI}, and on the response u_{FB} as well on the unbalance force f_U via $f_{RI} = S_U f_U$ as explained in Appendix A. Note that the entire influence of the foundation on the dynamics of the rotor is given by the force f_{FB} which can be calculated if the response u_{FB} is known.

There are two possibilities to determine u_{FB}: either by measurement or by calculation using a dynamic model of the foundation represented by F.

In the first case, u_{FB} is the difference of two measurements from two consecutive rundowns with and without known masses at known radii on given balance disks. The experimental effort is quite costly and therefore restricted, but the central interest is to determine the force f_{FB} in order to study its behaviour for various scenario of unbalances and variations of the bearing dynamics without performing numerous tests. In this case, the response u_{FB} needed to determine f_{FB} has to be calculated. If the transfer matrix F is known, the input/output equation (2) together with equation (12) enables the calculation of the foundation response as

$$u_{FB} = (F - C)^{-1} B G_{BI} f_{RI}. \tag{13}$$

Inserting this result into equation (12) leads to an expression of the foundation force

$$f_{FB} = F(F - C)^{-1} B G_{BI} f_{RI}. \tag{14}$$

This is the motivation for the estimation of the transfer matrix F which is usually done by substituting measurements u_{FB}^M for the response u_{FB} in equation (2) and (12) yielding

$$F u_{FB}^M = \underbrace{C u_{FB}^M + B G_{BI} f_{RI}}_{=: f_{FB}^M}. \tag{15}$$

Equation (15) has been derived and discussed in several previous papers (see for instance [2–4]). Two problems related to the estimation of F by using equation (15) are addressed here.

- Due to the condensation to eliminate the unknown internal dof the dynamic stiffness matrix, F depends non-linearly on the matrices of inertia, damping and stiffness of the foundation, and it also depends non-linearly on the excitation frequency ω.
- Since in general the bearing dynamics involved in the measurement u_{FB}^M differs from the description by the initial model B the resulting error of the force f_{FB}^M is transferred to the foundation model estimate.

The two points stated above and the solution proposed in this paper need further explanation.

2.1. THE MODEL ESTIMATION PROCESS

Mathematical modeling is always purpose orientated [6]. In the case discussed in this paper, the purpose is to estimate the influence of the foundation on the dynamics of the rotor. For this purpose, no physically interpretable model is necessary in order to model this influence. In most cases (see for instance [7, 8]), model estimation leads to a non-linear inverse problem. There are several possibilities to avoid solving a non-linear problem, as for instance to model the transfer matrix of the foundation by

$$F(\omega) := -\omega^2 M_i + j\omega D_i + K_i, \quad \omega \in \Omega_i \tag{16}$$

where the matrices M_i, D_i, K_i are allowed to vary over the frequency domain $\Omega = \Omega_1 \cup \cdots \cup \Omega_N$. The disadvantage of this modeling is the choice of the frequency intervals Ω_i which is related to the continuity at the interval bounds. In this paper, a frequency filter model (see Appendix C) is preferred. Although the resulting estimation problem is also linear and provides continuity over the entire frequency range, the disadvantage is the non-uniqueness of the solution. This is a typical expression for an ill-posed problem which can be turned into a well-posed problem by applying regularisation methods (see for instance [9]). The choice of an appropriate regularisation needs further investigation and is beyond the scope of this paper.

2.2. THE UNCERTAINTY IN THE BEARINGS MODEL

In general, at the resonance frequencies of the rotor connected to the bearings, the condition number of A_{RB} [see equation (8)] depends mainly on B. As demonstrated later in this paper according to the results of [3, 4] small errors in B relative to the real dynamics of the oil film contained in u_{FB}^M can lead to large changes in the force f_{FB}^M near those frequencies. Of course model errors due to the sensitivity of the force will be transferred to the estimation of the foundation transfer matrix F. As a consequence the calculation of the foundation force by substituting the estimate $\hat{F}$ of F in equation (14) can lead to erroneous predictions of the contribution of the elastic support to the rotor's dynamic behaviour. To avoid the propagation of such model errors several strategies have been suggested [1–5]. Lees and Friswell [3] suggested using only frequencies away from those resonance frequencies. This idea can be generalised by introducing frequency-dependent weighting factors within the estimation procedure. These weighting factors can be chosen to be minimum at the resonance frequencies and maximum at the anti-resonance frequencies of A_{RB}.

Of course the conclusions stated above are solutions to the problem of the bearings model sensitivity although this counter measure is not coherent with the philosophy of model estimation: to waste data means to waste information about the model. It is possible that the data near the resonance frequencies of the rotor connected to the bearings represent important information for the estimation of the foundation model. The resonance frequencies of A_{RB} might be different from those contained in u_{FB}^M. But since overlapping spectra cannot be excluded in general, the exclusion of such data within the foundation model estimation means the exclusion of data with a large signal-to-noise ratio.

2.3. THE PROPOSED SOLUTION

The solution of the problem of propagation of bearing model errors to errors of the foundation model estimate proposed in this paper consists of splitting this problem into model estimation and bearing model uncertainty. The method suggested here seeks to minimise the estimation error by estimating a transfer matrix H of minimum sensitivity with respect to the bearings model using an input/output equation of type

$$Hu_{FB}^M = f_H. \tag{17}$$

To find the most insensitive force f_H, it is suggested to determine a non-singular matrix P depending on the models of the rotor and the bearings such that

$$Pf_{FB}^M = PCu_{FB}^M + PBG_{BI}f_{RI} =: f_H \tag{18}$$

is of minimum sensitivity with respect to errors of the initial model of the bearings. In contrast to the input/output equation (15), a mismatch between the actual bearing dynamics contained in the data u_{FB}^M and the initial bearing model will lead to a minimum error in f_H only, because f_H was chosen to be of minimum sensitivity with respect to such errors. Thus, using equation (17), the estimate $\hat{H}$ of H will be robust with respect to errors to the initial bearing model. Of course, analytically the transfer matrix H is related to the foundation F by a transformation P via

$$PF = H \tag{19}$$

or

$$F = P^{-1}H. \tag{20}$$

Since P depends analytically on the bearing model B, H will vary for different bearing models. However, considering estimates $\hat{F}$, $\hat{H}$ of F, H, respectively equations (19) and (20) do hold only for P calculated at the initial bearing model, which has been used to estimate F and H. The bearing model error is included in the estimate $\hat{F}$ but in the estimate $P^{-1}\hat{H}$ the bearing model error is mainly contained in P via the dependency on B.

To clarify the difference between the use of $\hat{H}$ and $\hat{F}$, one can formulate an analytical expression for u_{FB} depending only on H and on P by inserting equation (20) into equation (13) yielding

$$u_{FB} = (H - PC)^{-1}PBG_{BI}f_{RI}. \tag{21}$$

Similarly, an analytical expression for the force f_{FB} can be obtained by inserting F from equation (20) into equation (15) yielding

$$f_{FB} = (C(H - PC)^{-1}P - I_{4m})BG_{RI}f_{RI}. \tag{22}$$

Note, that although at the initial bearing model equations (13) and (21) as well as equations (14) and (22) yield the same results they in general represent different functions of the bearing model when estimates based on an erroneous initial bearing model are substituted for H and F.

Inserting the estimate $\hat{H}$ into equation (21) the resulting expression can be understood as a function of B

$$b(B) := (\hat{H} - PC)^{-1}PBG_{BI}f_{RI}. \tag{23}$$

Similar to the function $b(B)$ one can define the function

$$a(B) := (\hat{F} - C)^{-1}BG_{BI}f_{RI}, \tag{24}$$

by inserting the estimate $\hat{F}$ into equation (13). Moreover, referring to equations (14) and (22), each of the functions defined in equations (24) and (23) allows the calculation of the foundation force as functions of the bearing model, depending on the estimate of F

$$\alpha(B) := [C(\hat{F} - C)^{-1} + I_{4m}]BG_{BI}f_{RI} = \hat{F}(\hat{F} - C)^{-1}BG_{BI}f_{RI} \tag{25}$$

or on the estimate of H

$$\beta(B) := [C(\hat{H} - PC)^{-1}P + I_{4m}]BG_{BI}f_{RI}. \tag{26}$$

One important point is that although, with reference to equation (15), at the initial bearing model, which has been used within the estimation process, $a \approx b \approx u^M_{FB}$ and $\alpha \approx \beta \approx f^M_{FB}$, a, b and α, β are generally different functions of B. Because of the additional dependency of b and β on P, they can only be expected to coincide with a and α when evaluated at the initial bearing model which has been used to estimate H and F. Whilst the error in the estimate of F will be transferred to errors in a and α the corresponding errors will be minimum using the robust estimate of H, i.e. the errors in b and β will be smaller.

Because $\hat{H}$ is robust whilst $\hat{F}$ is sensitive with respect to the initial bearing model the effect of an uncertain bearing model on the model estimation process is minimised.

The central point that remains to be explained is how to choose the transformation P such that $f_H = Pf^M_{FB}$ is of minimum sensitivity with respect to the initial bearing model. There are several possibilities, e.g. the requirement that P minimises the sensitivity matrix (see Appendix B) of the force f^M_{FB}. The method presented in this paper is based on a generalisation of the condensation process which in a particular case leads to the force f^M_{FB}.

Based on this generalisation, the next section derives an input/output equation of the general type (17) which is of minimum sensitivity with respect to the initial bearings model.

3. DEFINITION OF AN OPTIMUM INPUT/OUTPUT EQUATION

In this section, an input/output equation is derived in order to estimate a transfer matrix with a minimum sensitivity with respect to the bearing model. The underlying idea is a generalisation of the process which leads to the expression of the foundation force. The derivation of the force f_{FB} may be regarded as a nodal condensation process in which the internal dof of the rotor are eliminated. This observation makes it clear that the choice of reference dof is not unique although the complete elimination chosen in [3] is the most physically obvious. Thus, this conventional method to estimate the force can be generalised.

Essentially this generalisation consists of the choice of master and slave dof on the input side. This generalising aspect provides a redundancy which enables the definition of a force f_H which is insensitive with respect to changes in the bearing model.

To generalise the conventional method one may introduce the selecting matrix

$$T := [e_{i_1}, \ldots \ e_{i_{n+4m}}] \in \mathbb{R}^{(n+8m)\times(n+4m)}, \tag{27}$$

where $\{i_1, \ldots, i_{n+4m}\} \subset \mathscr{I} := \{1, \ldots, n+8m\}$, e_n denotes a unit vector of appropriate dimension containing zeros everywhere but in the nth place, and n and m are defined in Appendix A. In the process of dynamic condensation, this matrix selects the slave dof while the master dof are selected by the matrix

$$T^{\perp} := [e_{i_{n+4m+1}}, \ldots, e_{i_{n+8m}}], \tag{28}$$

where the two sets of ordered indices are disjoint and their union generates the complete set $\mathscr{I}$, i.e.

$$\{i_1, \ldots, i_{n+4m}\} \cap \{i_{n+4m+1}, \ldots, i_{n+8m}\} = \emptyset \tag{29}$$

$$\{i_1, \ldots, i_{n+4m}\} \cup \{i_{n+4m+1}, \ldots, i_{n+8m}\} = \mathscr{I}. \tag{30}$$

Writing equation (5) as

$$\underbrace{\begin{bmatrix} A_{RII} & A_{RIB} \\ A_{RBI} & A_{RBB}+B \\ 0 & -B \end{bmatrix}}_{=:W} \underbrace{\begin{pmatrix} u_{RI} \\ u_{RB} \end{pmatrix}}_{=:u_R} + \underbrace{\begin{bmatrix} 0 \\ -I_{4m} \\ I_{4m} \end{bmatrix}}_{=:S_1} B u_{FB} = \underbrace{\begin{pmatrix} f_{RI} \\ 0 \\ 0 \end{pmatrix}}_{=:h} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ I_{4m} \end{bmatrix}}_{=:S_2} f_{FB} \tag{31}$$

the multiplication of equation (31) by $T^{\top}$ yields

$$T^{\top} W u_R + T^{\top} S_1 B u_{FB} = T^{\top}(h - S_2 f_{FB}), \tag{32}$$

and the multiplication of equation (31) by $T^{\perp\top}$ leads to

$$T^{\perp\top} W u_R + T^{\perp\top} S_1 B u_{FB} = T^{\perp\top}(h - S_2 f_{FB}). \tag{33}$$

Providing $T^{\top} W$ is non-singular, equation (32) yields

$$u_R = (T^{\top} W)^{-1} T^{\top}(\underbrace{h - S_1 B u_{FB}}_{=:g} - S_2 f_{FB}), \tag{34}$$

which may be used to eliminate u_R in equation (33) from which one finds

$$f_H = P f_{FB}, \tag{35}$$

with the definitions

$$f_H := T^{\perp\top}[I_{n+8m} - W(T^\top W)^{-1}T^\top]g, \tag{36}$$

$$P := T^{\perp\top}[I_{n+8m} - W(T^\top W)^{-1}T^\top]S_2. \tag{37}$$

These are general expressions for the force f_H and the transformation matrix P. Of course the expressions of f_H and P depend on the choice of T. Indeed, the special case of choosing

$$T = [e_1, \dots, e_{n+4m}] = \begin{bmatrix} I_n \\ I_{4m} \\ 0 \end{bmatrix} \rightsquigarrow T^\perp = S_2 \tag{38}$$

leads via $T^\top S_2 = 0$ to $P = I_{4m}$ and thus $f_H = f_{FB}$. Note that for this choice of T, the matrix to be inverted becomes $T^\top W = A_{RB}$ which is found to be sensitive with respect to the bearing model. In this case equation (32) and (33) are equivalent to equations (6) and (7), respectively.

The basic idea of optimising the input/output equation is to choose the selecting matrix T in such a way that the force f_H has minimum sensitivity with respect to the model of the bearings. Depending on the computational capabilities this problem can be solved using different strategies as for instance:

- For a stochastically based simulation of bearing model errors choose that T which provides minimum variances of f_H.
- Choose that T which minimises some norm of the sensitivity matrix (see Appendix B) of the force f_H.
- Choose that T which provides a minimum condition number of the matrix $T^\top W$.

Of course T still has to select $n + 4n$ rows in order to ensure $T^\top W$ is a square matrix. But this can be done in several ways. Two criteria for defining the optimum are suggested.

3.1. CRITERIA FOR AN OPTIMUM CHOICE

As already noted, the original method is a special case of the general task to select $n + 4m$ rows in

$$W(\omega) = \begin{bmatrix} A_{RII}(\omega) & A_{RIB}(\omega) \\ A_{RBI}(\omega) & A_{RBB}(\omega) + B(\omega) \\ 0 & -B(\omega) \end{bmatrix} \in \mathbb{C}^{(n+8m)\times(n+4m)} \tag{39}$$

in order to achieve an invertible matrix providing f_H is of minimum sensitivity with respect to the bearings model represented by B. Of course this task is equivalent to the elimination of $4m$ rows in W. Since W only consists of known model quantities the task to determine the optimum selection can be done numerically. Since the sensitivity with respect to the bearing model is mainly due to the condition number of the matrix to be inverted one possible way to find the optimum selection is to solve the following minimax problem. Criterion 1: let $\mathcal{T}$ denote the set of all possible selecting matrices, i.e.

$$\mathcal{T} := \{[e_{i_1}, \dots, e_{i_{n+4m}}]: e_{i_k} \in \mathbb{R}^{n+8m}, \quad 1 \leqslant i_k \leqslant n + 8m, \quad \forall k = 1, \dots, n + 4m\}. \tag{40}$$

The optimum choice is given by

$$\min_{T\in\mathcal{T}} \max_{\omega\in\Omega} \operatorname{cond}(T^\top W(\omega)). \tag{41}$$

A low condition number is necessary but not sufficient in order to provide a low sensitivity of the force f_H with respect to B. Therefore a second numerical test can be applied using stochastic deviations in the bearing model. Let ΔK_i, $\Delta D_i \in [-1, 1]^2 \subset \mathbb{R}^2$ consist of uniformly distributed non-correlated random numbers with zero mean values and variances equal to one-third for all $i \in 1, \dots, 2m$. Refering to equation (A5), define

$$\Delta B_i \equiv \Delta B_i(s_i, r_i) := s_i \Delta K_i + j\omega r_i \Delta D_i, \tag{42}$$

where the positive scalars s_i, r_i control the magnitude of the random error of the ith bearing model. Thus, according to equation (A4) the error $\Delta B \equiv \Delta B(s, r)$ of the bearings model is well defined for $s := (s_1, \dots, s_{2m})^\top$ and $r := (r_1, \dots, r_{2m})^\top$. Regard the force $f_H = f(\omega, T, \Delta B)$ defined in equation (36) as a function f of the selecting matrix T and the bearing model error ΔB. For l random samples $\Delta B(k) = \Delta B(r(k), s(k)), k = 1, \dots, l$, calculate for each frequency $\omega \in \Omega$ the upper and lower bounds for the real and imaginary part of each component f_i, $i = 1, \dots, 4m$ of the force vector f, i.e.

$$f^{R}_{i\max}(\omega, T) := \max_{k=1,\dots,l} \mathrm{Re}\,(f_i(\omega, T, \Delta B(k))), \tag{43}$$

$$f^{R}_{i\min}(\omega, T) := \min_{k=1,\dots,l} \mathrm{Re}\,(f_i(\omega, T, \Delta B(k))), \tag{44}$$

$$f^{I}_{i\max}(\omega, T) := \max_{k=1,\dots,l} \mathrm{Im}(f_i(\omega, T, \Delta B(k))), \tag{45}$$

$$f^{I}_{i\min}(\omega, T) := \min_{k=1,\dots,l} \mathrm{Im}\,(f_i(\omega, T, \Delta B(k))). \tag{46}$$

Defining the force vectors

$$f_{\max}(\omega, T) := f^{R}_{\max}(\omega, T) + j \cdot f^{I}_{\max}(\omega, T), \tag{47}$$

$$f_{\min}(\omega, T) := f^{R}_{\min}(\omega, T) + j \cdot f^{I}_{\min}(\omega, T), \tag{48}$$

the second criterion can also be formulated as a minimax problem.
Criterion 2. The optimum selection is obtained from:

$$\min_{T \in \mathcal{T}} \max_{\omega \in \Omega} \sum_{i=1}^{4m} |f_{i\max}(\omega, T) - f_{i\min}(\omega, T)|. \tag{49}$$

Of course to find out the optimum $4m$ rows to be eliminated requires that all combinations without repetition of $n + 8m$ elements of order $4m$ are checked, i.e. the total number of different $(n + 4m)$-by-$(n + 4m)$ matrices $T^\top W$ is

$$\binom{n + 8m}{4m} := \frac{(n + 8m)!}{(4m)!(n + 4m)!} \tag{50}$$

which has to be done for all frequencies $\omega \in \Omega$. To avoid extensive computations two options are recommended.

- Restriction of the search to the most sensitive parts of the frequency domain Ω, which can be obtained from sensitivity analysis (see Appendix B) of the force f^{M}_{FB}.
- Performing the following routine leads to a suboptimum solution:

(i) Start with $i = 1$ and set $D := W$.
(ii) Take out each single row of D in turn to obtain $n + 8m - i$ matrices of format $(n + 8m - i)$-by-$(n + 4m)$.
(iii) Calculate the $n + 8m - i$ maximum condition numbers and store that row number which provides the smallest value.

(iv) Eliminate in D that row obtained from the previous step and overwrite D with the resulting matrix of format $(n + 8m - i)$-by-$(n + 4m)$.
(v) Stop if $i = 4m$ else increase $i \to i + 1$ and proceed with step (ii).

In contrast to the huge number of combinations given in equation (50) the algorithm defined above takes into account

$$\sum_{i=1}^{4m-1} (n + 8m - i) = (4m - 1)(n + 6m) \tag{51}$$

possibilities only. Depending on the computer capabilities available, the algorithm can be extended by considering all matrices resulting from elimination of $1 \leqslant k \leqslant 4m$ rows of D in step (ii).

4. A SIMPLE EXAMPLE

In this section, the method to find an input/output equation with minimum sensitivity with respect to changes in the uncertain interface model is demonstrated by a simple example.

4.1. THE TEST MODEL

The mounted machine is simulated by an Euler–Bernoulli beam which is discretised spatially using four beam elements (Fig. 2). The length of the beam is 2.3 m, the diameter is 0.1 m, the density is 7850 kg/m^3, and the Young's modulus is 2.1×10^{11} N/m^2. Each beam element is defined by two nodes. Each node has two dofs: one translational and one rotational. Thus the dimension of the beam model is 10.

The elastic interface system is modeled by two massless springs with stiffnesses $k_1 = 1.77 \times 10^8$ and $k_2 = 3.54 \times 10^8$ N/m, respectively. Thus, the matrix B of the initial interface model is given by

$$B = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{52}$$

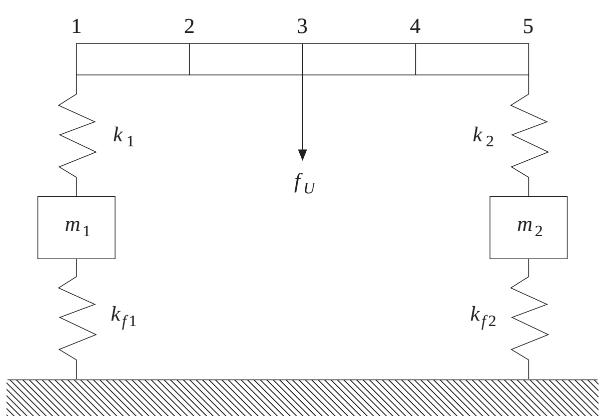

Figure 2. The simple test model.

Since only the translational dof of the first and last node of the beam are coupled to the two springs of the interface model, the number of connecting dofs is 2 and the number, n, of internal rotor dofs is 8.

The foundation is modeled by an unconnected pair of masses $m_1 = 90$, $m_2 = 135$ kg, and springs with stiffnesses $k_{f1} = k_{f2} = 1.77 \times 10^7$ N/m. This simple foundation model has no internal dof, i.e. the dimension $k = 0$. Refering to the notation in equations (A3), (A6) and (4) the dynamic stiffness matrix $A_F \equiv A_{FB} \equiv F$ is 2×2 and is given by

$$A_F(\omega) := 1.77 \times 10^7 I_2 - \omega^2 \begin{bmatrix} 90 & 0 \\ 0 & 135 \end{bmatrix}. \tag{53}$$

For the sake of simplicity, a single point excitation at the translation dof of node 3 has been considered. Thus, the force vector $f_{RI} \in \mathbb{R}^8$ has one non-zero component only

$$f_{RI} := f_U e_4 \in \mathbb{R}^8, \tag{54}$$

with the choice $f_U = 0.01\omega^2$. The responses are discretised over a frequency range between 0.5 and 250 Hz with equally spaced steps of 0.5 Hz, i.e. $M = 500$.

4.2. THE OPTIMUM CHOICE

To determine the optimum defined in the first criterion one has to check the condition numbers of all matrices which result from eliminating two rows of the matrix

$$W := \begin{bmatrix} A_{RII} & A_{RIB} \\ A_{RBI} & A_{RBB} + B \\ 0 & -B \end{bmatrix} \in \mathbb{R}^{12 \times 10}. \tag{55}$$

Refering to equation (50), this means calculating the maximum condition numbers of 66 matrices. The result is to eliminate rows 4 and 5. Thus, the optimum choice of a selecting matrix $T^{\perp}$ of the master dof is given by

$$T^{\perp} = [e_4, e_5] \in \mathbb{R}^{12 \times 2}, \tag{56}$$

which corresponds to the two dofs of node 3 (see Fig. 2). Note that the original method implies the use of $T^{\perp} = T_o^{\perp} := [e_{11}, e_{12}]$, which are the translational dof of the first and of the last node which couple the beam to the two springs representing the interface system.

The optimality of this choice is confirmed by the results of the second criterion. The computation has been done by adding uniformly distributed uncorrelated random errors with zero mean value to the stiffnesses k_i of the interface model, simulating a model variation of 50%:

$$k_i \to k_i + \frac{k_i}{2} \Delta k_i, \qquad i = 1, 2. \tag{57}$$

TABLE 1

The maximum difference in the force vector components

Component	f_{FB}	f_H
1	17.3	9.3×10^{-7}
2	13.8	3.6×10^{-7}

Here $\Delta k_i \in [-1, 1]$ are uncorrelated random numbers with expectation value $E\{\Delta k_i\} = 0$ and with variance $E\{\Delta k_i \Delta k_j\} = \frac{1}{3}\delta_{ij}$. For a size of $l = 500$ random samples, the upper and lower bounds $f_{\max}$, $f_{\min}$ and $f_{H\max}$, $f_{H\min}$ of the force $f_{FB} = f_{FB}(\omega, T_o, \Delta B)$ and of the force $f_H = f(\omega, T, \Delta B)$, have been calculated, respectively.

Table 1 shows the maximum difference of the bounds for the first and second component of the force f_{FB} and for the force f_H have been calculated. In contrast to the variation of the force f_{FB} of about 20 for the first and about 10 for the second component, the variation of the force f_H is for both components of the order 10^{-7}, which corresponds to the computational accuracy.

4.3. RESULTS OF MODEL ESTIMATION WITHOUT INTERFACE MODEL ERROR

In this Section, the results of the estimation of the filter parameters as well as some results concerning the robustness with respect to deviations of the interface stiffnesses are presented. As mentioned at the end of Appendix C, one needs a cut-off limit in order to decide the degree of the filter model that is adequate. Of course in practical applications, measurement error bounds or other regularisation information may serve to determine this limit (see for instance [9]). The inclusion of regularisation methods represents another problem and is not discussed in this paper. For the sake of simplicity, the computational accuracy is used to infer a moderate cut-off limit. Calculating the relative input and the relative output error using the 'true' foundation model defined in equation (53) to calculate the response u_{FB} and the excitation f_{FB} it was found that the maximum values of the errors defined in equation (C31) and (C32) with $f_H \to f_{FB}$ are approximately

$$e_I^o \approx 6.6 \times 10^{-7}, \tag{58}$$

$$e_O^o \approx 1.07 \times 10^{-5}, \tag{59}$$

where $(\ldots)^o$ indicates the use of the 'true' foundation model. The relative input error is about two orders more precise than the relative output error. To calculate the latter one has to invert the dynamic stiffness matrix which leads to the relative loss of accuracy. Amongst all possible model realisations with input and output powers $(n_i, n_o) \in \mathbb{N}^2$ which provides a maximum relative input and output error lower than 10^{-4} that with the smallest degree $p := n_i + n_o$ is choosen in the following simulations.

4.3.1. *Estimation of F*

At first the results of the model estimation are discussed using the original input/output equation defined in equation (15) and the initial interface model defined in equation (52). Solving the singular value problem (see Appendix C) for all input and output powers $(n_i, n_o) \in [0, 5]^2 \subset \mathbb{N}^2$ the maximum relative input and of the maximum relative output error as defined by equation (C31) and (C32) with $f_H \to f_{FB}$ have been calculated. Figure 3 shows model realisations ordered with respect to increasing sum $e_O + e_I$ of the errors. The cut-off limit of 10^{-4} for both errors leads to a model realisation $(n_i, n_o) = (0, 2)$ of minimum power. The values of the maximum relative errors are approximately

$$e_I \approx 1.7 \times 10^{-13}, \tag{60}$$

$$e_O \approx 3.7 \times 10^{-12}. \tag{61}$$

Comparing this result with the computational accuracy given by equation (58) and (59) the errors are about six orders lower. Besides an numerical noise of order 10^{-6}, the estimates of the filter matrices correspond to the matrices of the original model defined in equation (53). Using this filter model the estimate of F is, with reference to Appendix C, obtained by $\hat{F} = \tilde{B}^{-1}\tilde{A}$.

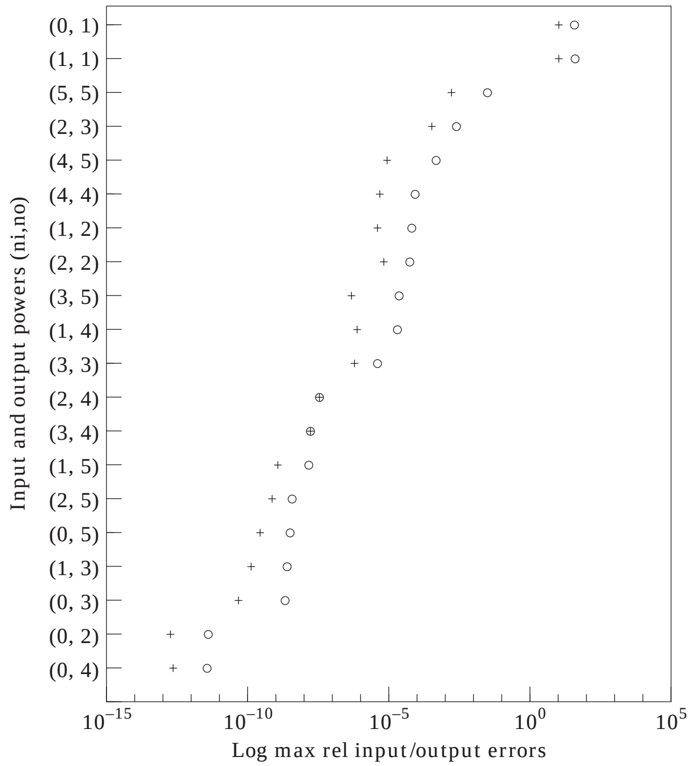

Figure 3. Maximum relative input(+) and output(○) errors for model realisations to estimate F without interface model error.

Before the results using the optimised input/output equation are presented some results concerning the robustness of the estimate using the original input/output equation are reported. Using the upper and lower bounds $f_{\max}$, $f_{\min}$ defined in the second criterion the upper and lower bounds of the relative input and output error have been calculated using the force vector

$$f_R(z) := \frac{f_{\max} + f_{\min}}{2} + \frac{f_{\max} - f_{\min}}{2} \odot z \in \mathbb{R}^2, \tag{62}$$

where $z \in [-1, 1]^2 \subset \mathbb{R}^2$ is a vector of uncorrelated uniformly distributed random numbers with zero mean values and variances $E\{z_i z_k\} = (1/3)\delta_{ik}$. The symbol $\odot$ denotes the Hadamard product [10], which is defined as the component-wise product of two vectors. The difference of the bounds of the maximum relative input and maximum relative output error respectively using $l = 500$ random samples $f_R(z_k)$, $k = 1, \ldots, l$ have been calculated. As expected, the variation of the interface stiffness of 50% leads via the associated variation of the force f_{FB} to very large variations of the relative errors. The variation of the maximum relative input error is of order 100 and the order of the variation of the maximum output error is approximately 10^4. One reason for this amplification lies in the deterioration of the nullity of the associated vector spaces span(X) (see Appendix C) which is related to the singular values of the matrix X. For the model realisation $(n_i, n_o) = (0, 2)$ it was found that due to the variation of the force f_{FB} the last two singular values are shifted up about five orders.

4.3.2. *Estimation of H*

Using the optimised input/output equation (17), the minimum values for both relative errors are available for an input power $n_i = 2$. The minimum values of the maximum relative input and the maximum relative output error are of order 10^{-10} and are related to a model degree of 25 which corresponds to the powers $(n_i, n_o) = (2, 23)$. Lower accuracy models with lower degree are available. For an input accuracy of about 6.5×10^{-5} a filter model of degree 12 is available with the powers $(n_i, n_o) = (4, 8)$. This model leads to an output accuracy of about 10^{-6}. Using this filter model the estimate of H, is with reference to Appendix C, given by $\hat{H} = \tilde{B}^{-1}\tilde{A}$.

The investigation of the robustness of the estimate $\tilde{H}$ has been done using the same random errors as in the case of the original method. The upper and lower bounds of the relative input and of the relative output errors due to random variation of the force f_H in between the bounds $f_{H\min}$, $f_{H\max}$ have been calculated. It was found that the variation is not small for every model realisation but for the chosen model with powers $(n_i, n_o) = (4, 8)$ the variation of the relative input error and the relative output error are of the same order

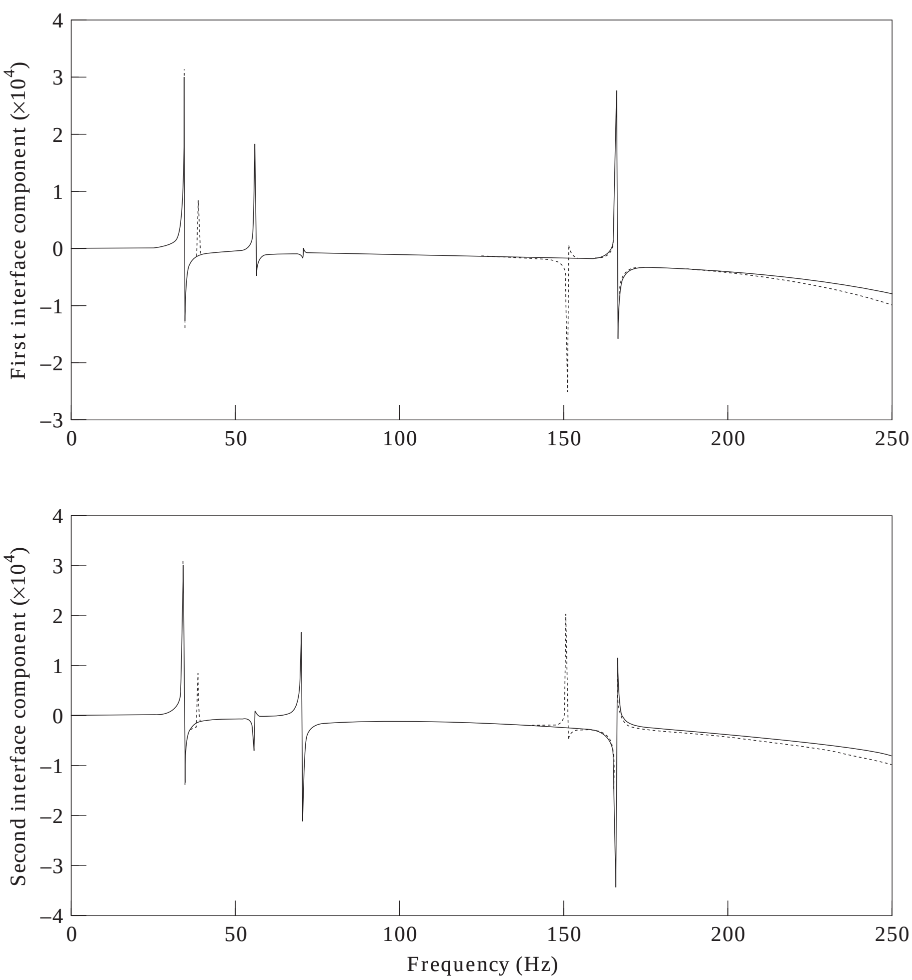

Figure 4. Foundation forces without (——) and with (· · · · ·) interface model error.

$\approx 10^{-4}$. Thus, compared with the order of variation 100 and 10^4 of the foundation estimate $\hat{F}$ the estimate $\hat{H}$ is robust with respect to errors of the interface model.

4.4. MODEL ESTIMATION WITH AN ERRONEOUS INTERFACE MODEL

To simulate a mismatch between the initial interface model and the real interface dynamics, the stiffnesses k_1, k_2 defined in equation (52) have been changed deterministically with a 1% increase of the first stiffness and a 1% decrease of the second stiffness of the initial interface model. Using these stiffnesses together with the foundation model defined in equation (53), the responses $u_{FB}(\omega)$, $\omega \in \Omega$ have been calculated to serve as measurements u_{FB}^M. Together with the original interface model defined in equation (52) they are then used in equation (15) to calculate $f_{FB}^M(\omega)$. Due to the sensitive dependency of f_{FB}^M on the interface model, a mismatch of only 1% leads to drastic changes (Fig. 4). In contrast to the 'true' foundation force, two new resonances occur due to the interface model error in f_{FB}. The foundation force f_{FB}^M and the response u_{FB}^M have been used to estimate the foundation transfer matrix F and the transfer matrix H, where the insensitive force f_H is given by the choice $T^\perp = [e_4, e_5]$ (see Section 4.2). Figure 5 shows the maximum relative input and output errors for some model realisations ordered with respect to increasing error sum $e_o + e_I$. In contrast to the estimate of F without error of the interface model (see Fig. 3) now a model realisation of $(n_i, n_o) = (4, 10)$ is necessary which provides an accuracy $(e_i, e_o) \approx (9.8, 1.2) \times 10^{-5}$. As expected, due to the minimum sensitivity property the estimation of H leads to the same input/output power $(n_i, n_o) = (4, 8)$ which provides an accuracy of $(e_i, e_o) \approx (1.2, 0.1) \times 10^{-5}$.

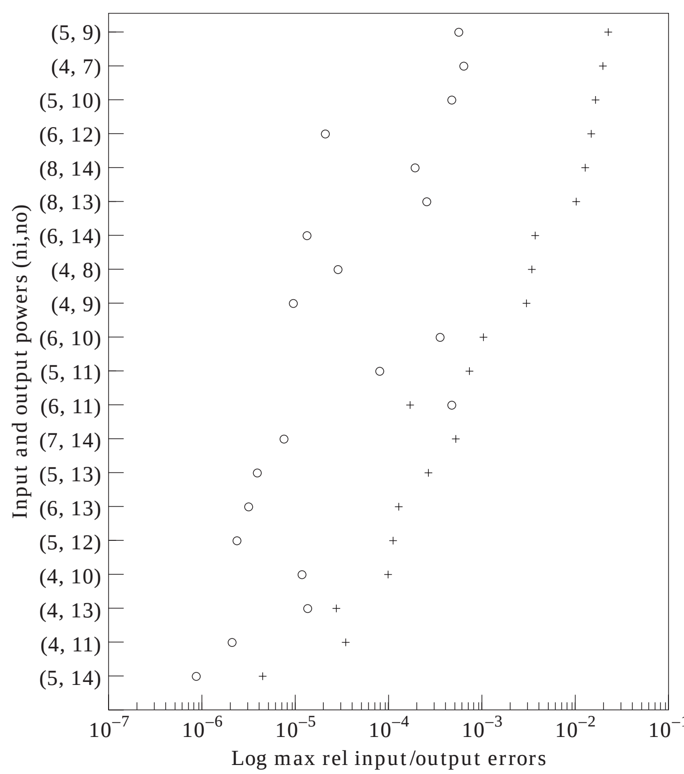

Figure 5. Maximum relative input(+) and output(○) errors for model realisations to estimate F using an erroneous interface model.

TABLE 2

The maximum errors relative to the 'true' values using four different error cases

Error case	Vector component	Response		Force	
		$a(B)$	$\alpha(B)$	$b(B)$	$\beta(B)$
1	1	1.3	1.8×10^{-6}	1.1	5×10^{-6}
	2	1.1	2.6×10^{-6}	0.94	4.5×10^{-6}
2	1	0.94	0.94	0.94	0.94
	2	0.94	0.94	1.25	1.25
3	1	1.4	2.7×10^{-3}	1.1	8.5×10^{-2}
	2	1.1	6.2×10^{-3}	0.93	7.1×10^{-2}
4	1	1.4	4×10^{-3}	1	8×10^{-2}
	2	1.1	1×10^{-2}	0.93	7×10^{-2}

To study the different behaviour of the functions a, b, α, β of B defined in equations (23) to (26) the initial interface model has been changed deterministically by adding errors $\Delta k_i = k_i p_i/100$ to the stiffness k_i, $i = 1, 2$ defined in equation (52). Four different cases have been investigated:

$$(p_1, p_2) \in \{(1, -1), (0, 0), (5, 10), (-5, 5)\}. \tag{63}$$

For each case the resulting matrix B has been inserted in equations (24) to (26). To compare the results the 'true' foundation responses and forces have been calculated using the foundation model defined in equation (53) and the modified interface model of the four test cases. In Table 2 the maximum relative output and input errors are listed for each of the four cases using the 'true' responses and forces in equations (C31) and (C32). Note that the second case corresponds to the initial bearing model used to estimate F and H together with the response calculated using the bearing model of the first case. Thus, for the second case the errors are equal for a, b and α, β respectively. Comparing the errors of different cases, the error of a and b are of the same order ≈ 1, whilst the errors of α and β become maximum for case 2 and minimum for case 1. Although the estimation of H has been done with an erroneous bearing model, the prediction of the responses and the forces calculated with different bearing models are more reliable due to the minimum sensitivity property of the estimate of H.

5. CONCLUSION

The study of the influence of an elastic support on the dynamics of the mounted system is of great interest in a wide range of practical applications. Often the elastic support consists of an elastic interface system and the grounded system of the foundation. In this case the contribution of the support on the dynamics of the mounted system is given by the foundation force at the interface locations. Generally, this force cannot be measured but it can be estimated depending on the models of the three subsystems involved. When only the dynamic model of the mounted system is reliable, the model of the interface system is uncertain, and a model of the foundation is completely unknown, the original method tries to estimate a model of the foundation using measured responses and estimated forces at the interface points. Since the estimate of the force is sensitive with respect to mismatches between the interface dynamics contained in the measured responses and the initial interface model, the resulting force errors are transferred to errors of the foundation model estimate. Using this erroneous model estimate within the force estimation to study the effect of the support on the dynamics of the mounted system leads to non-reliable predictions.

The method presented in this paper allows the minimisation of the effect of an uncertain interface model on the model estimates. The method produces an optimised choice of the input/output equation that is used to estimate a transfer matrix. In contrast to the estimate of the transfer matrix of the foundation this transfer matrix is robust with respect to deviations of the interface model. Analytically, both transfer matrices are related by a transformation which depends on the model of the mounted system and of the interface model. The advantage of this method is that the robust transfer matrix estimate rather than the erroneous foundation transfer matrix estimate can be used to estimate the foundation force for studying the effect of variations of the interface model on the dynamics of the mounted system. The use of the robust transfer matrix estimate leads to an improved accuracy of the predicted responses and forces of the foundation at the interface locations because the variations of the interface model alter only the transformation matrix whilst the robust transfer matrix estimate is still valid for those variations.

Strategies to define the optimised choice of the input/output equation have been discussed, and two criteria for an optimum choice have been suggested.

To avoid a non-linear model estimation, the transfer matrix has been modeled by a filter in the frequency domain which is sufficient to simulate the contribution of the elastic support on the dynamics of the mounted system. This modeling strategy has the disadvantage that with increasing model degree, the equation error can be made arbitrarily small. Because this error should not be smaller than the accuracy of the data, further investigation should allow the choice of the model degree to be related to data errors.

ACKNOWLEDGEMENTS

The authors thank Nuclear Electric Ltd. and Magnox plc for funding a research project to develop a method to estimate the influence of the elastic support on the dynamics of the rotor.

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APPENDIX A: MODEL AND DATA

In order to determine a model for the foundation, run-down data are available. These are the run-down response $u_{FB}^M(\omega)$ of the foundation at the bearings caused by an unbalance force $f_U(\omega)$ on the rotor, both given at discrete frequencies $\omega \in \Omega := \{\omega_1, \ldots, \omega_M\}$. It is assumed that aside from two unbalances per shaft no other (relevant) forces act on the system. Thus, for a rotor consisting of m shafts the vector f_U is $2m$-dimensional. An m-shaft rotor is usually connected to the foundation via $2m$ bearings. Since the data consists of the transverse and the vertical movement at each bearing the dimension of the vector u_{FB} is $4m$. Let $n, k \in \mathbb{N}$ be the dimensions of the internal part of the rotor model and of the internal part of the foundation model, respectively. The input/output models of the rotor, the bearings and the foundation can be represented using the (linearised) dynamic stiffness matrices which are partitioned according to the internal dof and to those connected to the bearings.

rotor

$$A_R u_R = f_R \Leftrightarrow \begin{bmatrix} A_{RII} & A_{RIB} \\ A_{RBI} & A_{RBB} \end{bmatrix} \begin{pmatrix} u_{RI} \\ u_{RB} \end{pmatrix} = \begin{pmatrix} f_{RI} \\ -f_{FB} \end{pmatrix} \in \mathbb{C}^{n+4m}, \tag{A1}$$

bearings

$$A_B u_B = f_B \Leftrightarrow \begin{bmatrix} B & -B \\ -B & B \end{bmatrix} \begin{pmatrix} u_{RB} \\ u_{FB} \end{pmatrix} = \begin{pmatrix} f_{FB} \\ -f_{FB} \end{pmatrix} \in \mathbb{C}^{8m}, \tag{A2}$$

foundation

$$A_F u_F = f_F \Leftrightarrow \begin{bmatrix} A_{FBB} & A_{FBI} \\ A_{FIB} & A_{FII} \end{bmatrix} \begin{pmatrix} u_{FB} \\ u_{FI} \end{pmatrix} = \begin{pmatrix} f_{FB} \\ 0 \end{pmatrix} \in \mathbb{C}^{k+4m}. \tag{A3}$$

In equations (A1) to (A3), the two rules of model synthesis are involved:

- kinematic compatibility requires the responses at the connection dof to be the same;
- due to Newton's principle of *actio et reactio* the forces at the connection dof differ only in sign.

In addition, the special structure of the bearings model is already taken into account. It can be shown that the matrix B is block diagonal, i.e.

$$B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_{2m} \end{bmatrix}, \tag{A4}$$

which in general the matrix $B_i \in \mathbb{C}^{2\times 2}$ for the ith bearing is given by

$$B_i = K_i + j\omega D_i, \quad i \in \{1, \ldots, 2m\}, \tag{A5}$$

with $j := \sqrt{-1}$. The stiffness matrices and the damping matrices, K_i, $D_i \in \mathbb{R}^{2\times 2}$, result from linearisation and are in general non-symmetric and non-singular, and depend on the excitation frequency. Connecting the three systems represented by their dynamic stiffness matrices leads to the input/output model for the entire system:

$$\begin{bmatrix} A_{RII} & A_{RIB} & 0 & 0 \\ A_{RBI} & A_{RBB} + B & -B & 0 \\ 0 & -B & B + A_{FBB} & A_{FBI} \\ 0 & 0 & A_{FIB} & A_{FII} \end{bmatrix} \begin{pmatrix} u_{RI} \\ u_{RB} \\ u_{FB} \\ u_{FI} \end{pmatrix} = \begin{pmatrix} f_{RI} \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^{n+k+8m}. \tag{A6}$$

Of course the non-zero entries of the force vector f_{RI} are the $2m$ components of the vector of unbalance forces f_U, i.e. there exists a so-called selecting matrix $S_U \in \mathbb{R}^{n\times 2m}$ such that

$$f_{RI} = S_U f_U, \quad \text{or} \quad S_U^\top f_{RI} = f_U. \tag{A7}$$

In general, a matrix $S \in \mathbb{R}^{n\times m}$ is called a selecting matrix if $S^\top S = I_m$ and

$$SS^\top = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \tag{A8}$$

where in general I_n denotes the n-dimensional unit matrix. The dynamic stiffness matrix $A_{FII} \in \mathbb{C}^{k\times k}$ representing the internal part of the foundation is non-singular in the frequency range of interest because the entire system is grounded.

APPENDIX B: SENSITIVITY MATRIX OF THE FOUNDATION FORCE

The estimation of the transfer matrix $F(\omega)$ which maps the response $u_{FB}(\omega)$ of the foundation at the bearings to the force $f_{FB}(\omega)$ of the foundation at the bearings depends sensitively on the precise values of the dynamic stiffness matrix $B(\omega)$ of the bearings via the force f_{FB}^M defined in equation (A5) which can be written in the form

$$f_{FB}^M(\omega) = -B(\omega)u_{FB}^M(\omega) + [0, B(\omega)]A_{RB}^{-1}(\omega)\begin{pmatrix} f_{RI}(\omega) \\ B(\omega)u_{FB}(\omega) \end{pmatrix}. \tag{B1}$$

To calculate the sensitivity matrix of the force f_{FB}^M with respect to the bearing model B at frequency ω, the following parameterisation of the dynamic stiffness matrix B is used

$$\mathscr{B}(Q, \omega) := \sum_{(i,j)\in\mathscr{K}} E_{ij} b_{ij}(\omega) q_{ij}, \tag{B2}$$

where $b_{ij}(\omega)$ denotes the element in row i and column j of the initial dynamic stiffness matrix of the bearings at frequency ω, and $E_{ij} := e_i e_j^\top$ is the elementary matrix consisting of zeros except the entry in row i and column j which is equal to unity. The set $\mathscr{K}$ is a subset of the cartesian product of the index set $\{1, \ldots, 4m\}$ with itself and is due to the block diagonal structure of B [see equation (A4)]. When all the dimensionless parameters

$q_{ij} \equiv (Q)_{ij}$ in equation (B2) are equal to unity (denoted by $Q = \mathbb{1}$), the dynamic stiffness matrix of the initial model results, i.e.

$$\mathscr{B}(Q)_{Q=1} = B. \tag{B3}$$

This parameterisation enables the calculation of an approximation for deviations of the force due to deviations of the initial bearing model. For sufficiently small deviations $\Delta B(\omega) = \mathscr{B}(\Delta Q, \omega)$ of the initial dynamic stiffness matrix of the bearings, the Taylor series approach leads to a foundation force at frequency ω

$$f_{FB}^M(\mathbb{1} + \Delta Q, \omega) = f_{FB}^M(\mathbb{1}, \omega) + \sum_{(i,j)\in\mathscr{K}} f_{FBij}^M(\mathbb{1}, \omega)\Delta q_{ij} + O(\|\Delta Q\|^2), \tag{B4}$$

where f_{FBij}^M denotes the derivative of f_{FB}^M with respect to q_{ij}. Since the difference $f_{FB}^M(\mathbb{1} + \Delta Q, \omega) - f_{FB}^M(\mathbb{1}, \omega)$ of both force vectors is approximately the linear combination represented by the second term on the right-hand side of equation (B4) the matrix

$$F_o := [f_{FB11}, f_{FB12}, \ldots, f_{FB4m(4m-1)}, f_{FB4m4m}] \tag{B5}$$

containing the series of the vectors $f_{FBij}\forall(i,j)\in\mathscr{K}$ of the derivatives of f_{FB} with respect to q_{ij} is called sensitivity matrix. Refering to equation (B1), the derivative of f_{FB} with respect to the element q_{ij} in row i and column j of the parameter matrix Q is

$$\begin{aligned}
f_{FBij} &= -b_{ij}E_{ij}u_{FB}^M + [0, b_{ij}E_{ij}]A_{RB}^{-1}\begin{pmatrix} f_{RI} \\ Bu_{FB} \end{pmatrix} + [0, B]A_{RB}^{-1}\begin{pmatrix} 0 \\ b_{ij}E_{ij}u_{FB} \end{pmatrix} \\
&\quad - [0, B]A_{RB}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & b_{ij}E_{ij} \end{bmatrix} A_{RB}^{-1}\begin{pmatrix} f_{RI} \\ Bu_{FB} \end{pmatrix} \\
&= \Bigg(-e_i u_{FBj}^M + e_i\underbrace{(0, e_j^\top)A_{RB}^{-1}\begin{pmatrix} f_{RI} \\ Bu_{FB} \end{pmatrix}}_{=:a_j} + \underbrace{[0, B]A_{RB}^{-1}\begin{pmatrix} 0 \\ e_i \end{pmatrix}}_{=:b_i} u_{FBj}^M - b_i a_j\Bigg) b_{ij} && \text{(B6)} \\
&= \underbrace{(b_i - e_i)}_{=:v_i}\underbrace{(u_{FBj}^M - a_j)b_{ij}}_{=:r_{ij}} && \text{(B7)} \\
&= v_i r_{ij}. && \text{(B8)}
\end{aligned}$$

A measure of the sensitivity of the force f_{FB}^M with respect to changes of the element b_{ij} of the initial bearing model at frequency ω can be defined by

$$s_{ij}(\omega) := \|f_{FBij}^M(\omega)\| = \|v_i(\omega)\||r_{ij}(\omega)|. \tag{B9}$$

All vectors f_{FBij}^M can be written in matrix form as

$$F_o = [v_1, \ldots, v_{4m}]K \begin{bmatrix} r_{11} & & & & 0 \\ & r_{12} & & & \\ & & \ddots & & \\ & & & r_{4m(4m-1)} & \\ 0 & & & & r_{4m4m} \end{bmatrix}, \tag{B10}$$

where the constant matrix K sorts the vectors v_i in proper order.

APPENDIX C: MODELING IN THE FREQUENCY DOMAIN: THE FILTER MODEL

In general, a filter model in the frequency domain [11] for an input/output equation of type (17) has the form

$$\underbrace{\left(\sum_{i=0}^{n_o}(j\omega)^i A_i\right)}_{=:\tilde{A}(\omega)} u_{FB}^M(\omega) = \underbrace{\left(\sum_{k=0}^{n_i}(j\omega)^k B_k\right)}_{=:\tilde{B}(\omega)} f_H(\omega) \tag{C1}$$

The output and input powers n_o, n_i respectively, and the matrices $(A_i)_{i=0,\dots,n_o}$, $(B_k)_{k=0,\dots,n_i}$ are called filter parameters and have to be estimated. Of course the minimum of $\det[\tilde{A}(\omega)]$ and of $\det[\tilde{B}(\omega)]$ are related to the resonance and anti-resonance frequencies of the transfer matrix H respectively, because

$$H = \tilde{B}^{-1}\tilde{A}. \tag{C2}$$

In order to estimate the filter parameters, the least squares method can be applied by minimising the equation error. Defining the ith partial equation residual as

$$v_E(i) := \tilde{A}(\omega_i) u_{FB}^M(\omega_i) - \tilde{B}(\omega_i) f_H(\omega_i) \tag{C3}$$

the cost function to be minimized is given by

$$J_E := \sum_{i=1}^{M} v_E^\dagger(i) W_E(i) v_E(i), \tag{C4}$$

where $W_E(i)$ denotes a weighting matrix for the ith equation residual. Assuming $W_E(i) = I_{4m}$ for all $i = 1, \dots, M$, the filter equation (C1) can be extended for M excitation frequencies as

$$\left(\sum_{i=0}^{n_o} A_i U \Lambda^i\right) = \left(\sum_{k=0}^{n_i} B_i Z \Lambda^k\right), \tag{C5}$$

where U, Z and Λ are defined by

$$U := [u_{FB}^M(\omega_1), \dots, u_{FB}^M(\omega_M)], \tag{C6}$$

$$Z := [f_H(\omega_1), \dots, f_H(\omega_M)], \tag{C7}$$

$$\Lambda := j\begin{bmatrix}\omega_1 & & 0\\ & \ddots & \\ 0 & & \omega_M\end{bmatrix}. \tag{C8}$$

The solution of the minimisation problem (C4) is equivalent to the normal solution of equation (C5) which can be rewritten as

$$\underbrace{[A_{n_o}, \ldots, A_0, -B_{n_i}, \ldots, -B_0]}_{=:V} \underbrace{\begin{bmatrix} U\Lambda^{n_o} \\ \vdots \\ U \\ Z\Lambda^{n_i} \\ \vdots \\ Z \end{bmatrix}}_{=:Y} = 0 \tag{C9}$$

Because the filter parameters represented by the matrix $V \in \mathbb{R}^{4m \times 4m(n_o+n_i+2)}$ are real-valued, equation (C9) must be satisfied for the real and imaginary parts of the matrix $Y \in \mathbb{C}^{4m(n_o+n_i+2) \times M}$, which finally yields

$$V[\mathrm{Re}\,(Y), \mathrm{Im}(Y)] := VX = 0. \tag{C10}$$

This problem does not lead to a unique solution for the filter parameters. Indeed, for any arbitrary non-singular matrix C,

$$C\tilde{A}u_{FB} = C\tilde{B}f_H \tag{C11}$$

is also a solution. Since one is interested in the product $\tilde{B}^{-1}\tilde{A} =: H$ (or its inverse) only this final result is of interest and this product is unique.

As a necessary and a sufficient condition for a full-rank solution V of equation (C10) the matrix $X \in \mathbb{R}^{4m(n_o+n_i+2) \times 2M}$ has to have a rank deficiency of $4m$, i.e.

$$\mathrm{rank}(X) = 4m(n_o + n_i + 1). \tag{C12}$$

Of course, this problem has to be treated numerically. The rank decision is usually made by looking to the singular values $\gamma(n_o, n_i) \in \mathbb{R}^{4m(n_o+n_i+2)}$ of the matrix $X = X(n_o, n_i)$. Because one cannot expect to achieve zero rather than relative small singular values one has to define a cut-off limit. This is due to the fact that the equation error (C4) can be made arbitrary small by increasing the degree $p := n_o + n_i$ of the filter model. The same situation occurs if one looks to the maximum relative input error

$$e_I := \max_{i=1,\ldots,M} \frac{\|\tilde{B}^{-1}(\omega_i)\tilde{A}(\omega_i)u_{FB}^M(\omega_i) - f_H(\omega_i)\|}{\|f_H(\omega_i)\|}, \tag{C13}$$

or to the maximum relative output error

$$e_O := \max_{i=1,\ldots,M} \frac{\|\tilde{A}^{-1}(\omega_i)\tilde{B}(\omega_i)f_H(\omega_i) - u_{FB}^M(\omega_i)\|}{\|u_{FB}^M(\omega_i)\|}. \tag{C14}$$

With increasing degree p the errors e_I and e_O can be made arbitrary small.