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THE EVALUATION OF ROTOR IMBALANCE IN FLEXIBLY MOUNTED MACHINES

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A method is presented to determine the state of imbalance of a rotating machine by using the measured pedestal vibration. The only requirements of the procedure are a good numerical model for the rotor and an approximate model for the bearing behaviour. No assumptions are made concerning the operational mode shape of the rotor and the influence of the supporting structure is included in a consistent manner. For simplicity the analysis is presented in a single plane orthogonal to the rotor axis, but no difficulty is foreseen in extending the method to two planes. Examples are given for a two-bearing system.

1. INTRODUCTION

Methods of balancing can be categorized into two groups, the influence coefficient method which only requires the assumption of linearity of both the machine and measuring system, and modal balancing which in addition requires a knowledge of the modal properties of the machine. The former of these approaches has the attraction of requiring less *a priori* knowledge of the system and techniques have been well developed to make optimum use of redundant information [1]. The approach does however suffer from the significant disadvantage of requiring a number of test runs on site. For machinery with a high commercial output, this is a significant disadvantage.

Modal approaches require fewer test runs but, as mentioned, require prior knowledge of the machine. Two methods have been proposed in recent years [2, 3] which offer the prospect of balancing without test runs. Gnielka [2] used prior knowledge of mode shapes and modal masses and compared results to those from a numerical model of the machine. The work of Krodkiewski et al. [3] has similar requirements and seeks to detect changes in imbalance from running data. Both these approaches place reliance on the model of the machine. In a real machine, however, the modal properties of the machine may be significantly influenced by the properties of the structure on which it is supported, and this has proved extremely difficult to model [4]. Numerical models of rotating machinery have been used to great effect over a number of years [5, 6], and their accuracy and range of effectiveness have been steadily developing. The principle limitation is now thought to be the unknown effects of the foundation and recently several authors have addressed this issue [7–9].

In principle, if a numerical model of a machine were sufficiently reliable, rotor imbalance could be derived directly from the measured vibration levels at the bearing pedestals. In this paper it is shown how rotor imbalance may be derived from measured data by using an accurate model of the rotor and some knowledge of the bearing behaviour. No knowledge is required of the supporting structure: this is represented as stiffness and mass

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matrices whose coefficients are determined as a part of the calculation. It is shown that good estimates for the imbalance at each plane may be derived, but information concerning the phase of the imbalance is not derived in this paper as it is strongly dependent on the quality of the bearing model. Furthermore it is shown that additional data concerning model parameters may be introduced into the solution to improve accuracy of results.

2. BEARING FORCES

In this work, the rotor is considered to be adequately modelled. It is assumed that the free–free modal properties of the complete rotor train may be calculated from the model. The free–free modes of the component rotors may be measured by suspending the rotor in slings and can be used to verify the model. Knowledge of these free–free modes is simply one way of representing the dynamics of the rotor itself; mode shapes and frequencies are readily transformed into stiffness and mass matrices of the rotor. There is additional advantage in this approach since rotors are often identical amongst a class of machines whereas supporting structures are extremely variable [4].

Given the rotor model, the motion of the rotor is the result of two sets of forces. Firstly, there is a set of imbalance forces which can arise anywhere along the rotor, but over the running speed range of a machine, the resultant can be represented as a limited number of modal contributions, which in principle may be different along the rotor. Secondly, forces arise in the bearings and the time dependent part of these forces arises as a consequence of the imbalance forces. At any point x along the rotor, the displacement y_r is given by

$$y_r(x) = \int_0^L G(\omega, x, x')F(x')\,\mathrm{d}x', \tag{1}$$

where $F(x')$ represents the force per unit length along the rotor, and $G(\omega, x, x')$ is the Green's function of the rotor, representing the response at point x arising from a unit force at x'. (A list of notation is given in Appendix 2.) This function depends on frequency, and one has the standard result

$$G(\omega, x, x') = \sum_{j=1}^{\infty} \frac{\psi_j^*(x)\psi_j(x')}{\omega_j^2 - \omega^2}, \tag{2}$$

where the mode shapes have been normalized to the rotor mass, i.e.,

$$\int_0^L \psi_j^*(x)\rho(x)\psi_j(x)\,\mathrm{d}x = 1. \tag{3}$$

Here $\rho(x)$ is the mass per unit length of the rotor at x, ω_j is the jth natural frequency of the rotor (free–free), ψ_j represents the corresponding mode shape and * denotes the complex conjugate transpose. The frequencies and mode shapes may be estimated by using a discrete approximation via a finite element model of the rotor. In the above representation, no allowance has been made for damping within the rotor. There would be little difficulty including such a term but in practice damping from the bearings and supporting structure will normally be dominant in turbo-machinery. In the present calculation the damping is neglected for the sake of clarity.

The forces acting on the rotor are of two types, the imbalance at various locations which are acting at m locations $x_{e1}, \ldots, x_{em}$, which are taken as a set of pre-defined locations.

Note that for a finite range of running speeds, the effective inbalance distribution may be represented as a summation over the balance planes [10]. The unknown bearing reaction forces act at the n bearing locations $x_{b1}, \ldots x_{bn}$. In many cases $n = m$, but separate symbols are retained for generality and clarity. The bearing forces $F_{b1}, \ldots, F_{bn}$ have yet to be determined. The total force acting on the rotor becomes

$$F(x, \omega) = \sum_{i=1}^{m} \omega^2 e_i \, \delta(x - x_{ei}) + \sum_{i=1}^{n} F_{bi} \, \delta(x - x_{bi}), \tag{4}$$

where δ is the Dirac delta function, and e_i is the imbalance (i.e., the product of imbalance mass and radius) at location x_{ei}. Inserting this expression into equation (1) yields an equation for the shaft motion. At each of the bearings of the system, the force can be related to the shaft displacement within the bearings. Writing the displacements of the shaft and bearing pedestal as y_r and y_p respectively, and combining equations (1) and (4), gives an expression for the displacement of the shaft at any point x:

$$y_r(x) = \sum_{i=1}^{m} G(\omega, x, x_{ei}) \omega^2 e_i + \sum_{i=1}^{\mathrm{n}} G(\omega, x, x_{bi}) F_{bi}. \tag{5}$$

The bearing forces in this equation may be expressed as

$$F_{bi} = -k_i \, (y_{ri} - y_{pi}), \tag{6}$$

where k_i is the stiffness of the ith bearing, $y_{ri} = y_r(x_{bi})$ and y_{pi} is the corresponding pedestal displacement. Note that this is the force acting on the rotor—with an equal and opposite force acting on the foundation structure. A speed dependent model including damping terms is readily included by rewriting equation (6) as

$$F_{bi}(\omega) = -(k_i(\omega) + j\omega c_i(\omega)) \, (y_{ri}(\omega) - y_{pi}(\omega)), \tag{7}$$

where all the frequency dependent terms have been included explicitly. Inclusion of these terms presents no difficulties to the model. In the numerical examples presented below constant stiffness values of the pedestal displacements y_{pi} are assumed; then a set of simultaneous equations may be formed for the shaft displacements at the bearing location by setting $x = x_{b1}, \ldots, x_{bn}$. Hence

$$y_{rj} = \sum_{i=1}^{m} G(\omega, x_{bj}, x_{ei}) \omega^2 e_i + \sum_{i=1}^{n} k_i \, (y_{ri} - y_{pi}) G(\omega, x_{bj}, x_{bi}). \tag{8}$$

Thus, on determining the displacements y_{pi} at each bearing location the forces acting on each of the bearing pedestals may be determined if all of the imbalance components are known. Multiplying equation (5) by k_j gives

$$k_j \, y_{rj} = k_j \sum_{i=1}^{m} G(\omega, x_{bj}, x_{ei}) \omega^2 e_i + k_j \sum_{i=1}^{n} G(\omega, x_{bj}, x_{bi}) F_{bi}. \tag{9}$$

One can now eliminate y_{rj} by using equations (6) and (9), and, after a little rearranging, obtain an equation for each plane of vibration at each bearing:

$$-F_{bj} + k_j \, y_{pj} = k_j \sum_{i=1}^{m} G(\omega, x_{bj}, x_{ei}) \omega^2 e_i + k_j \sum_{i=1}^{n} G(\omega, x_{bj}, x_{bi}) F_{bi}. \tag{10}$$

From this, introducing the abbreviated notation $G_{pq} = G(\omega, x_{bp}, x_{bq})$ and $G_{pe_q} = G(\omega, x_{bp}, x_{eq})$, one may write

$$\begin{Bmatrix} F_1 \\ \vdots \\ F_n \end{Bmatrix} = [a]^{-1} \begin{Bmatrix} k_1 y_{p1} - k_1 \sum_{i=1}^{m} G_{1e_i} \omega^2 e_i \\ \vdots \\ k_n y_{pn} - k_n \sum_{i=1}^{m} G_{ne_i} \omega^2 e_i \end{Bmatrix}, \tag{11}$$

where

$$a_{ij} = \delta_{ij} + k_i G_{ij} \tag{12}$$

In equation (12) i and j range over all bearings and δ_{ij} is the Kronecker delta

Note that the expression for the forces exerted on the rotor and the reaction on the bearing pedestal are a function of the bearing stiffnesses, but are independent of the foundation stiffness. The foundation will, of course, strongly influence, the pedestal vibration y_p, but this is measured.

A full discussion of this approach and the conditioning of the matrix $[a]$ has been given by the authors [11, 12].

3. ANALYSIS OF MULTI-BEARING ROTOR

In previous studies [11], the imbalance was considered to be known. In the present work, however, the imbalance in each of the pre-set balancing planes may be represented as an unknown vector $\{e\} = \{e_1, \ldots, e_n\}^{\mathrm{T}}$. By using this notation, equation (11) may be re-written in the vector form

$$\{F\} = [a]^{-1} [k_B]\{y_p\} - \omega^2[a]^{-1}[k_B][G_{be}]\{e\}, \tag{13}$$

where $[k_B] = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, $\{y_p\} = \{y_{p1}, y_{p2}, \ldots, y_{p_n}\}^{\mathrm{T}}$ and $[G_{be}]_{ij} = G_{ie_j}$. Equation (13) may be written more conveniently as

$$\{F\} = [p]\{y_p\} - [q]\{e\}, \tag{14}$$

where $[p] = [a]^{-1}[k_B]$ and $[q] = \omega^2[a]^{-1}[k_B][G_{be}]$. The dimension of the vector $\{F\}$ is the number of bearings (times two if both perpendicular directions are considered). The vector $\{F\}$ represents the force acting on the rotor from the bearing reaction; hence the force acting on the supporting structure is just $-\{F\}$. The matrix $[G_{be}]$ represents the relationship between balance planes and bearings and it is just a subset of the frequency dependent free–free rotor description, as described by equation (2).

Let the dynamic behaviour of the supporting structure be represented by the undetermined stiffness and mass matrices $[K]$ and $[M]$. A damping matrix may also be introduced, but since the damping of rotating machinery is usually dominated by the behaviour of the bearing oil film (which is easily included in the analysis), the damping matrix may be neglected. By using these matrices, a further vector equation may be written to express the force vector as

$$-\{F\} = [K]\{y_p\} - \omega^2[M]\{y_p\}, \tag{15}$$

and an equation is formed at each frequency measured. Note that equation (15) limits the number of modes in the foundation to equal the number of bearings. In the case of a foundation with many modes within the running frequency range, the structural matrices should vary. This may be treated by subdivision of the frequency range, resulting in a larger Least Squares (LS) problem, but no additional difficulty in principle. Since the elements of the matrices are unknown, it is convenient to re-write this equation in the form

$$-\{F\} = [w(\omega)]\{v\}, \tag{16}$$

where $[w]$ contains all reference to $\{y_p\}$, and the vector $\{v\}$ contains the elements of matrices $[K]$ and $[M]$. The ordering of this vector is arbitrary, but for the two bearing, single direction case one can make the choice $\{v\} = \{k_{11}\ k_{22}\ k_{12}\ m_{11}\ m_{22}\ m_{12}\}^{\mathrm{T}}$, where k_{ij} represents the (i, j)th element of $[K]$. It has been assumed that both $[K]$ and $[M]$ are symmetric so that $k_{ij} = k_{ji}$ and $m_{ij} = m_{ji}$. In this case $[w(\omega)]$ is the 2×6 matrix

$$[w(\omega)] = \begin{bmatrix} y_{p1}(\omega) & 0 & y_{p2}(\omega) & -\omega^2 y_{p1}(\omega) & 0 & -\omega^2 y_{p2}(\omega) \\ 0 & y_{p2}(\omega) & y_{p1}(\omega) & 0 & -\omega^2 y_{p2}(\omega) & -\omega^2 y_{p1}(\omega) \end{bmatrix}. \tag{17}$$

Equating the two expressions at frequency yields the equation

$$-[w(\omega)]\{v\} = [p]\{y_p\} - [q(\omega)]\{e\}. \tag{18}$$

Writing these equations at each frequency monitored gives a single equation,

$$[W]\{v\} + [Q]\{e\} = \{P\}, \tag{19}$$

where the matrices $[Q]$ and $[W]$ and the vector $\{P\}$ are formed by the addition of each corresponding sub-matrix as

$$[Q] = \begin{bmatrix} q(\omega_1) \\ q(\omega_2) \\ \vdots \\ q(\omega_N) \end{bmatrix}, \quad [W] = \begin{bmatrix} w(\omega_1) \\ w(\omega_2) \\ \vdots \\ w(\omega_N) \end{bmatrix} \quad \text{and} \quad \{P\} = \begin{bmatrix} [p(\omega_1)]\{y_p(\omega_1)\} \\ [p(\omega_2)]\,[y_p(\omega_2)\} \\ \vdots \\ [p(\omega_N)]\,[y_p(\omega_N)\} \end{bmatrix},$$

where ω_i is the ith of N frequency points at which displacement is measured.

Equation (19) may be written in the form

$$[W \quad Q]\begin{bmatrix} v \\ e \end{bmatrix} = \{P\}. \tag{20}$$

This represents an over-specified problem, provided sufficient frequencies are measured, for which one may find the Moore–Penrose pseudo-inverse giving the least squares solution as

$$\begin{Bmatrix} v \\ e \end{Bmatrix} = \begin{bmatrix} W^{\mathrm{T}}W & W^{\mathrm{T}}Q \\ Q^{\mathrm{T}}W & Q^{\mathrm{T}}Q \end{bmatrix}^{-1} \begin{bmatrix} W^{\mathrm{T}} \\ Q^{\mathrm{T}} \end{bmatrix}\{P\}. \tag{21}$$

In many practical situations there may not be sufficient information to determine all parameters, but this may be overcome by the introduction of other physical information, as outlined below, or by regularization of the problem by other means. A convenient approach to this is by the use of Singular Value Decomposition (SVD), and limiting the number of singular values used for the matrix inversion. Use of this technique was not necessary for the examples in the paper but the method can be found in reference [13].

The direct solution method has been used in the present calculations in which $[W]$ and $[Q]$ are both real. In practice these matrices will usually be complex, and in that case equation (21) must be divided into real and imaginary parts. The unknown parameters are real and are given by the solution to the equation

$$\begin{bmatrix} \mathrm{Re}\,[W] & \mathrm{Re}\,[Q] \\ \mathrm{Im}\,[W] & \mathrm{Im}\,[Q] \end{bmatrix} \begin{Bmatrix} v \\ \mathrm{e} \end{Bmatrix} = \begin{Bmatrix} \mathrm{Re}\,(P) \\ \mathrm{Im}\,(P) \end{Bmatrix}. \tag{22}$$

This solution may easily be modified to take account of any knowledge of the structure which may be available, such as one or more of the direct stiffness terms. Such knowledge may arise from modelling or prior tests. If part of $[K]$ is known then one column of $[W]$ is removed and $[W]$ becomes $[W_R]$. If the column removed is a vector $\{W_j\}$ then

$$\begin{Bmatrix} v \\ e \end{Bmatrix} = \begin{bmatrix} W_R^{\mathrm{T}} W_R & W_R^{\mathrm{T}} Q \\ Q^{\mathrm{T}} W_R & Q^{\mathrm{T}} Q \end{bmatrix} \begin{bmatrix} W_R^{\mathrm{T}} \\ Q^{\mathrm{T}} \end{bmatrix} \{\{P\} - v_J\, W_J\}, \tag{23}$$

where v_J is the value of the known parameter. Hence in principle, all desired parameters can be determined. Care is required to optimize the conditioning of the regression matrix in equations (21), (22) or (23). This is achieved by appropriate scaling of parameters. A scaling for the spatial displacement may be derived from that given by a unit imbalance at unit scaled frequency with the reference stiffness. Hence $x_r = 1/m_r$. After inserting these equations, the original equation becomes

$$\left[\frac{K}{m_r\,\omega_r^2}\right]\begin{Bmatrix} \dfrac{x}{x_r} \end{Bmatrix} - \left(\frac{\omega}{\omega_r}\right)^2 \left[\frac{M}{m_r}\right]\begin{Bmatrix} \dfrac{x}{x_r} \end{Bmatrix} = \begin{Bmatrix} \dfrac{F}{m_r\,\omega_r^2\,x_r} \end{Bmatrix}, \tag{24}$$

and with these appropriately chosen values, each of the terms in this equation is of order of unity. The choice of reference frequency ω_r is the only arbitrary choice. This choice is made in such a way as to minimize the condition number of the matrix to be inverted.

Equation (19) remains valid for the study of two orthogonal directions. If both orthogonal directions are considered, and there is damping present in the bearing model, $[W]$, $[Q]$ and $\{P\}$ all become complex. The parameters to be identified remain real. The vector $\{v\}$, neglecting damping, is now $\{k_{11yy}\ k_{11zz}\ k_{22yy}\ k_{22zz}\ k_{11yz}\ k_{12yz}\ k_{22yz}\ k_{12yy}\ k_{12zz}\ k_{12yz}\}^{\mathrm{T}}$ with corresponding mass terms and each component of $w(\omega)$ is given by the relation

$$[w(\omega)] = [[\hat{w}] - \omega^2[\hat{w}]],$$

where

$$[\hat{w}] = \begin{bmatrix} y_{p1} & 0 & 0 & 0 & z_{p1} & 0 & 0 & y_{p2} & 0 & z_{p2} \\ 0 & z_{p1} & 0 & 0 & y_{p1} & y_{p2} & 0 & 0 & z_{p2} & 0 \\ 0 & 0 & y_{p2} & 0 & 0 & z_{p1} & z_{p2} & y_{p1} & 0 & 0 \\ 0 & 0 & 0 & z_{p2} & 0 & 0 & y_{p2} & 0 & z_{p1} & y_{p1} \end{bmatrix}, \tag{25}$$

and z is the transverse direction orthogonal to y. The imbalance vector now becomes a four component vector $\{e\} = \{e_{1y}\ e_{1z}\ e_{2y}\ e_{2z}\}^{\mathrm{T}}$ where in each of the balance planes there are two component to the imbalance, referred to a reference shaft marker position. The resulting equation is solved as before (equation (22)).

Figure 1. Two-bearing rotor.

4. A SIMPLE EXAMPLE

The example considered is illustrated in Figure 1. A simple rotor 4 m long having mass 1450 kg is mounted on two bearings of stiffness 177 and 354 MN/m respectively. These bearings are independently supported on foundations which are each represented by a single mass and stiffness. These two support stiffnesses are both set at 177 MN/m whilst the two pedestal masses are 90 and 135 kg respectively. With these parameters the natural frequencies of the rotor alone are those of two rigid modes, 67 and 183 Hz, whilst the frequencies of the rotor mounted on the bearings but with rigid pedestals are 29 and 100 Hz.

The bearing stiffnesses were held constant over the frequency range. It is recognized that this is unrealistic, but the simple model suffices to illustrate the important features of the method. Table 1 shows the range of tests considered. The first test uses data from 120 frequency points over the range of 0–120 Hz, which contains the first two modes. Figure 2 shows the displacement of the two bearings. In the next six cases studied, the effects of restricting the frequency range and the number of data points is examined in the presence of noise. Noise is added as a factor times the high frequency response of the system. Figure 3 shows the noisy data for the two bearings for test case 3.

Since the method requires input of the bearing stiffnesses, tests 9–12 consider the effects of an error in these stiffnesses. Perfect data is assumed in case 9 and a noise factor of 5%

TABLE 1

Summary of cases considered for two-bearing example

Case	Frequency range (Hz)	No. of frequency steps	Noise (relative to high freq. level)	Bearing error (%)
1	120	120	0	0
1a	120	120	0	0
2	120	120	0·02	0
3	120	120	0·05	0
4	120	50	0·05	0
5	80	50	0·05	0
6	50	50	0·05	0
7	80	30	0·05	0
8	120	120	0·2	0
9	120	120	0	10
10	120	120	0·05	10
11	120	120	0·05	20
12	120	120	0·02	50
12a	120	120	0·02	50
12b	120	120	0·05	50

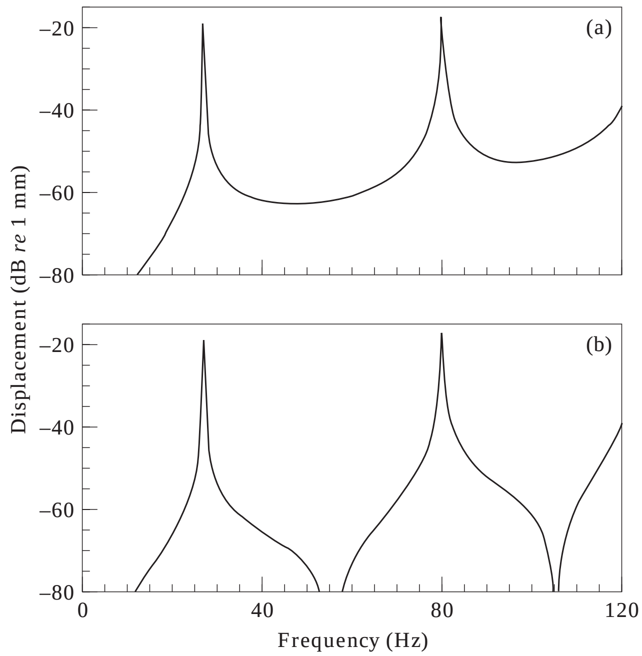

Figure 2. Absolute pedestal displacements for case 1. (a), Bearing 1; (b), bearing 2.

is used in the remaining cases. The error in the stiffness of both the bearings is 10, 20 and 50% respectively.

5. RESULTS

Table 2 shows the results for the cases given in Table 1. Each calculation using Gaussian random noise was the average of twenty independent calculations. The table shows the means and standard deviations for each of the parameters. The convergence of the statistics was checked with some longer runs of one of the examples. It was established that, whilst the results are not fully converged for 20 samples, the figures are representative and considered sufficiently accurate for the present study. In cases 1–12 shown in Table 1, one bearing is of the same stiffness as the pedestal, whilst the other one is double the stiffness. This is a demanding example for the study as the treatment of the bearing forces becomes a better approximation for a soft foundation. The study of machines in which the supporting structure is more flexible than the bearing is covered by cases 1a, 12a and 12b. This is representative of many large modern turbo-generators.

In case 1, the identification calculation is carried out with no noise present. No difficulty is experienced in the correct identification of both the elastic coefficients of the model and the unbalance at the pre-determined balance planes. Cases 2 and 3 cover the same frequency range, covering two natural frequencies, and both cases show excellent treatment of the noisy signals, particularly in establishing the imbalance.

Case 4 utilizes fewer frequency steps but retains the same frequency range. The effect of the noise produces larger errors in the estimated parameters, although the results are still acceptable. Case 8 considers the highest level of noise; mean imbalance predictions, however, remain within 5% of the exact result, whilst the standard deviation is within 30%.

Note that this level of noise is unlikely in practice. Nevertheless, a machine balance on this would reduce vibration substantially.

As the frequency range is reduced to 50 Hz, case 6, the accuracy of the model parameters becomes unacceptable. Note that this reduced frequency range covers only one of the natural frequencies of the system. It is worthy of note, however, that the standard deviation of the imbalance is only 5% of the true level and the mean is 20% in error. Cases 5 and 7 cover a wider frequency range but with many fewer frequencies. It is not surprising that the noise degrades the estimate of imbalance.

It was considered important to examine the sensitivity of identified parameters to uncertainties in the bearing stiffnesses. These are highly uncertain in a real machine, partly as a result of variations in static loading. Therefore, identification of model parameters and imbalance was carried out with varying error levels in the assumed bearing stiffness. Cases 10–12 have bearing stiffness errors of 10, 20 and 50% respectively and the resulting errors are modest, the largest being 20%. Note that with this same noise and error levels for a more flexible support structure (with a stiffness of 29·5 MN/m) the error in imbalance reduces to about 6% as shown in cases 1a, 12a and 12b.

Whilst the standard deviation of structural parameter, particularly the masses are large in many cases, smaller deviations are observed in the imbalance estimates. It is this feature which shows some promise for technique.

6. DISCUSSION

It has been established in section 5 that the method presented in this paper for the identification of imbalance is appropriate for use on machinery in which the bearing oil films are at least as stiff as the foundation over the speed range of interest. This is because the estimation of bearing force levels is insensitive to assumed levels of bearing

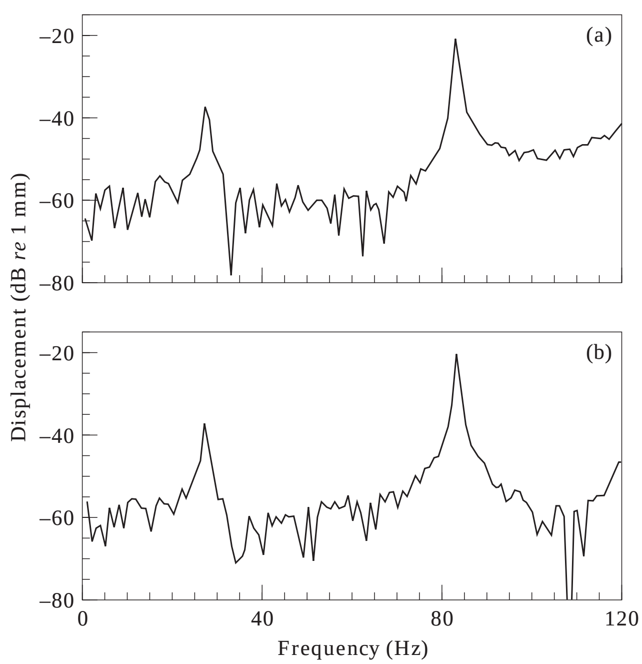

Figure 3. As Figure 2 but for case 8—20% noise added.

TABLE 2

Results for two-bearing example: estimated parameters; results of 20 runs of each case—standard deviations in brackets

Case	$k_{11} \times 10^{-8}$ (N/m)	$k_{22} \times 10^{-8}$ (N/m)	$k_{12} \times 10^{-8}$ (N/m)	m_{11} (kg)	m_{22} (kg)	m_{12} (kg)	e_1 (kgmm)	e_2 (kgmm)
1	1·77(0)	1·77(0)	0(0)	95(0)	135(0)	0(0)	2(0)	3(0)
1a	0·29(0)	0·29(0)	0(0)	95(0)	135(0)	0(0)	2(0)	3(0)
2	1·75	1·74	0·025	85	125	9·9	2·01	3·02
	(0·15)	(0·16)	(0·15)	(59)	(62)	(61)	(0·07)	(0·09)
3	1·70	1·69	0·07	68	108	28	2·02	3·03
	(0·33)	(0·34)	(0·34)	(128)	(136)	(134)	(0·18)	(0·21)
4	1·58	1·59	0·06	35	81	35	1·93	2·96
	(0·52)	(0·55)	(0·53)	(208)	(225)	(214)	(0·29)	(0·30)
5	0·56	0·57	0·74	23	48	683	2·47	2·76
	(0·12)	(0·11)	(0·11)	(65)	(70)	(67)	(2·51)	(2·54)
6	0·91	0·86	0·88	−65	−106	186	2·54	2·43
	(0·06)	(0·04)	(0·04)	(216)	(141)	(118)	(0·12)	(0·13)
7	0·18	0·23	0·38	−155	−122	511	0·40	0·78
	(0·21)	(0·27)	(0·23)	(116)	(139)	(122)	(3·09)	(3·09)
8	1·87	1·85	−0·11	154	188	−30	2·10	3·11
	(0·41)	(0·41)	(0·41)	(164)	(166)	(163)	(0·70)	(0·76)
9	1·62	1·62	0·03	70	108	15	1·87	2·81
	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
10	1·84	1·85	−0·19	158	199	−71	1·88	2·79
	(0·52)	(0·54)	(0·53)	(210)	(212)	(209)	(0·24)	(0·16)
11	1·46	1·46	0·11	34	68	47	1·78	2·67
	(0·53)	(0·56)	(0·55)	(212)	(221)	(215)	(0·24)	(0·22)
12	1·31	1·32	0·1	21	59	45	1·53	2·34
	(0·75)	(0·78)	(0·77)	(298)	(310)	(304)	(0·22)	(0·26)
12a	0·29	0·29	−0·007	102	133	−8·3	1·89	2·92
	(0·002)	(0·002)	(0·002)	(2·7)	(3·1)	(2·9)	(0·01)	(0·01)
12b	0·29	0·29	−0·007	103	134	−8·7	1·89	2·91
	(0·007)	(0·008)	(0·007)	(9·5)	(11)	(10)	(0·03)	(0·03)

stiffness over most of the frequency range. Whilst this assumption covers a range of turbo-machinery of practical interest, it is not valid in all circumstances. For machines mounted on very stiff supports, the methodology may still be employed, but either an accurate assessment of bearing stiffness must be available, or use must be made of absolute rotor displacement at the bearings. If these absolute measurements are available, a bearing model is not required for the analysis. Appendix 1 outlines the modifications to the calculation required in this case.

No damping has been included in the studies in this paper. The inclusion of damping presents no real difficulties, but for an analysis in a single plane, the prediction of phase will be examined in a subsequent paper. For a fully self-consistent analysis, it is believed that consideration and measurements are necessary in both directions perpendicular to the rotor axis. This analysis follows logically from the methods presented above, but a full study of the sensitivity of the case is beyond the scope of the present paper.

A notable feature of the method is that in the presence of noise the imbalance estimates are more accurate than the mass and stiffness parameters. The mass parameters are the terms most prone to errors in the estimation procedure. Clearly this behaviour is related to the form of the equation errors which are minimized, and this phenomena will be investigated more fully in later work. Since imbalance estimation is an important practical problem, the accuracy demonstrated above indicates that a useful technique has been outlined.

7. CONCLUSIONS

A method has been presented to derive imbalance components by using measured data and a model of the rotor only. The analysis is insensitive to bearing parameters for machines with flexible supports. The approach is easily generalized to two directions. Although stiffness and mass terms show moderate sensitivity to uncertainties, the method has been shown to produce imbalance estimates which are insensitive to measurement noise. Data is required covering as many modes as there are balancing planes.

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REFERENCES

1. J. Drechsler 1980 *Institution of Mechanical Engineers Conference on Vibrations in Rotating Machinery*, *Cambridge*, *UK*, 65–70. Processing surplus information in computer aided balancing of large flexible rotors.
2. P. Gnielka 1983 *Journal of Sound and Vibration* **90**, 157–172. Modal balancing of flexible rotors without test runs: an experimental investigation.
3. J. M. Krodkiewski, J. Ding and N. Zhang 1994 *Journal of Sound and Vibration* **169**, 685–698. Identification of unbalance change using a non-linear mathematical model for rotor bearing systems.
4. A. W. Lees and I. C. Simpson 1983 *Institution of Mechanical Engineers Conference*, C6/83, London, UK. The dynamics of turbo-alternator foundations.
5. T. H. McCloskey and M. L. Adams 1992 *Institute of Mechanical Engineers Conference on Vibrations in Rotating Machinery*, Paper C432/137, 239–250, Bath, UK. Troubleshooting power plant rotating machinery vibration using computational techniques: case histories.
6. C. Frigeri, G. A. Zanetta and A. Vallini 1988 *Institution of Mechanical Engineers Conference on Vibrations in Rotating Machinery*, Paper C302/88, 395–404, Edinburgh, UK. Some in-field experiences of non-repeatable behaviour in the dynamics of rotating machinery.
7. A. W. Lees 1988 *Institution of Mechanical Engineers Conference on Vibrations in Rotating Machinery*, *Paper C*306/065, 209–215, Edinburgh, UK. The least squares method applied to investigating rotor/foundation interactions.
8. G. A. Zanetta 1992 *Institution of Mechanical Engineers Conference on Vibrations in Rotating Machinery*, Paper C432/092, 173–182, Bath, UK. Identification methods in the dynamics of turbo-generator rotors.
9. N. S. Feng and E. J. Hahn 1995 *Mechanical Systems and Signal Processing* **9**, 243–256. Including foundation effects on the vibration behaviour of rotating machinery.
10. W. Kellenburger 1972 *Transactions of the American Society of Mechanical Engineers*, *Journal of Engineering for Industry* **94**, 584–560. Should a flexible rotor be balanced in N or N + 2 planes?
11. A. W. Lees and M. I. Friswell 1996 *International Conference on Identification in Engineering Systems* 793–803, Swansea, UK. Estimation of forces exerted on machine foundations.
12. M. I. Friswell, A. W. Lees and M. G. Smart 1996 *NAFEMS Conference on Structural Dynamics Modelling*: *Test Analysis and Correlation*, 461–472, Cumbria, UK. Model updating techniques applied to turbogenerators mounted on flexible foundations.
13. G. Golub and C. Van Loan 1989 *Matrix Computation*. John Hopkins University Press.

APPENDIX 1: INCLUSION OF ROTOR ABSOLUTE MOTION MEASUREMENT

From equation (8), the rotor displacement at any bearing position is given by

$$y_{rj} = \sum_{i=1}^{m} G_{je_i}\, \omega^2 e_i + \sum_{i=1}^{n} G_{ji}\, F_{bi}. \tag{A1}$$

Then for frequencies away from the free–free resonances of the rotor, the forces may be written as

$$\{F\} = [\alpha(\omega)]^{-1}\{y_r - [G_{be}(\omega)]\omega^2\{e\}\}, \tag{A2}$$

where

$$[\alpha(\omega)] = \begin{bmatrix} G(\omega, x_{b1}, x_{b1}) & \cdots & G(\omega, x_{b1}, x_{bn}) \\ \vdots & \vdots & \vdots \\ G(\omega, x_{bn}, x_{b1}) & \cdots & G(\omega, x_{bn}, x_{bn}) \end{bmatrix}. \tag{A3}$$

Hence one may write an equation corresponding to equation (14) of the main text:

$$\{F\} = [\alpha(\omega)]^{-1}\{y_r\} - [q']\{e\}. \tag{A4}$$

Note that q' differs from q of equation (14) only insofar as the matrix $[a']$ which replaces the $[a]$ of equation (12) does not have the delta function term present: i.e.,

$$[q'] = \omega^2[\alpha]^{-1}[G_{be}]. \tag{A5}$$

Then the analysis presented in the paper continues and equation (19) is replaced by

$$[W]\{v\} + [Q]\{e\} = \{P_r\}, \tag{A6}$$

where

$$\{P_r\} = \begin{Bmatrix} [\alpha(\omega_1)]^{-1}\{y_r(\omega_1)\} \\ [\alpha(\omega_2)]^{-1}\{y_r(\omega_2)\} \\ \vdots \\ [\alpha(\omega_N)]^{-1}\{y_r(\omega_N)\} \end{Bmatrix}. \tag{A7}$$

In equation (A6) the matrices $[W]$ is unchanged whilst $[Q]$ is modified as outlined in equation (A5). The solution for the simple case with no constraints is

$$\begin{Bmatrix} v \\ e \end{Bmatrix} = \begin{bmatrix} W^T W & W^T Q \\ Q^T W & Q^T Q \end{bmatrix}\{P_r\}. \tag{A8}$$

APPENDIX 2: NOTATION

x	axial distance along rotor
y	displacement orthogonal to rotor
z	second orthogonal direction
L	length of rotor
G	rotor Green function
F	force on rotor
$\{e\}$	imbalance
n	number of bearings
m	number of imbalance planes
k	bearing stiffness
c	bearing damping
$[a]$	force sensitivity matrix at a given frequency
G_{ij}	discrete representation of G
$[K]$	foundation stiffness matrix
$[M]$	foundation mass matrix
$[w]$	frequency dependent matrix of displacements
$[v]$	vector of unknown components
$[W]$	concatenation of $[w]$

$[q]$	imbalance influence matrix defined in equation (14)
$[Q]$	concatenation of $[q]$
$[p]$	force influence matrix
$\{P\}$	generalized force vector defined in equation (19)

Greek

$\delta(\ )$	Dirac delta function
δ_{ij}	Kronecker delta
ω	angular frequency
ρ	rotor mass per unit length
ψ	mode shape (free–free)

Subscripts

i, j	counters
bi	bearing i
ei	imbalance plane i
r	rotor
p	pedestal