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MODEL REDUCTION USING DYNAMIC AND ITERATED IRS TECHNIQUES

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Static or Guyan reduction is widely used to reduce the number of degrees of freedom in a finite element model but it is exact only at zero frequency. The Improved Reduced System (IRS) method makes some allowance for the inertia terms and produces a reduced model which more accurately estimates the modal model of the full system. In this paper the IRS method is extended by obtaining the equivalent transformation based on dynamic rather than static reduction. An iterative algorithm, based on the IRS method, is also described. On convergence this algorithm provides a reduced model which reproduces a subset of the modal model of the full system. The iterative version of the IRS method based on dynamic reduction is also described.

1. INTRODUCTION

Model reduction, whereby the number of degrees of freedom in a model is reduced, is applied to large finite element models to give faster computation of the natural frequencies and mode shapes of a structure. Model reduction also has a role to play in experimental modal analysis, since the reduced mass and stiffness matrices may also be used to compare the analytical and experimental models by using orthogonality checks. The transformation inherent in the model reduction schemes may also be used to expand the measured mode shapes to the full size of the finite element model, and these mode shapes may then be used in test analysis correlation or model updating exercises.

One of the oldest and most popular reduction methods is static or Guyan reduction [1]. In this process the inertia terms associated with the discarded degrees of freedom are neglected. However, while exact for a static model, when applied to a dynamic model the reduced model generated is not exact and often lacks the required accuracy. O'Callaghan [2] proposed a modified method which he called the Improved Reduced System (IRS) method. In this approach an extra term is added to the static reduction transformation to make some allowance for the inertia forces. This extra term allows the modal vectors of interest in the full model to be approximated more accurately but relies on the statically reduced model.

In this paper the IRS method is extended in two ways: by using the transformation from dynamic reduction instead of static reduction as the basic transformation; and by introducing an iterative scheme in which the corrective term is generated iteratively using

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the current best estimate of the reduced model. The convergence of the natural frequencies of the reduced model to those of the full model is examined. The method is demonstrated by using a 15-degree-of-freedom discrete mass–spring system, a 90-degree-of-freedom model of a clamped slotted plate and a 96-degree-of-freedom plane frame.

2. THE STANDARD IMPROVED REDUCED SYSTEM (IRS) METHOD

Possibly the most popular and certainly the simplest reduction method is static reduction, introduced by Guyan [1]. The state and force vectors, **x** and **f**, and the mass and stiffness matrices, **M** and **K**, are split into sub vectors and matrices relating to the master degrees of freedom, which are retained, and the slave degrees of freedom, which are eliminated. If no force is applied to the slave degrees of freedom and the damping is negligible, the equation of motion of the structure becomes

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_m \\ \ddot{\mathbf{x}}_s \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_m \\ \mathbf{x}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_m \\ \mathbf{0} \end{Bmatrix}. \tag{1}$$

The subscripts m and s relate to the master and slave co-ordinates respectively. Neglecting the inertia terms for the second set of equations gives

$$\mathbf{K}_{sm}\mathbf{x}_m + \mathbf{K}_{ss}\mathbf{x}_s = \mathbf{0}, \tag{2}$$

which may be used to eliminate the slave degrees of freedom so that

$$\begin{Bmatrix} \mathbf{x}_m \\ \mathbf{x}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} \end{bmatrix} \{\mathbf{x}_m\} = \mathbf{T}_s\mathbf{x}_m, \tag{3}$$

where $\mathbf{T}_s$ denotes the static transformation between the full state vector and the master co-ordinates. The reduced mass and stiffness matrices are then given by

$$\mathbf{M}_R = \mathbf{T}_S^{\mathrm{T}}\mathbf{M}\mathbf{T}_S, \qquad \mathbf{K}_R = \mathbf{T}_S^{\mathrm{T}}\mathbf{K}\mathbf{T}_S, \tag{4}$$

where $\mathbf{M}_R$ and $\mathbf{K}_R$ are the reduced mass and stiffness matrices. Note that any frequency response functions generated by the reduced matrices in equations (4) are exact only at zero frequency. As the excitation frequency increases the inertia terms neglected in equation (1) become more significant.

O'Callaghan [2] improved the static reduction method by introducing a technique known as the Improved Reduced System (IRS) method. The method perturbs the transformation from the static case by including the inertia terms as pseudo-static forces. Obviously, it is impossible to emulate the behaviour of a full system with a reduced model and every reduction transformation sacrifices accuracy for speed in some way. O'Callaghan's technique [2] results in a reduced system which matches the low frequency resonances of the full system better than static reduction. However, the IRS reduced stiffness matrix will be stiffer than the Guyan reduced matrix and the reduced mass matrix is less suitable for othogonality checks than the reduced mass matrix from Guyan reduction [3]. Gordis [3] generated the transformation for the standard IRS method by using a binomial series expansion in frequency. This derivation is summarized here for comparison with the derivation of the dynamic IRS transformation. From equation (1), for sinusodial excitation with frequency ω,

$$[\mathbf{K}_{ss} - \omega^2\mathbf{M}_{ss}]\mathbf{x}_s = -[\mathbf{K}_{sm} - \omega^2\mathbf{M}_{sm}]\mathbf{x}_m. \tag{5}$$

Rearranging this equation and using the binomial theorem gives

$$\mathbf{x}_s = -[\mathbf{K}_{ss} - \omega^2\mathbf{M}_{ss}]^{-1}[\mathbf{K}_{sm} - \omega^2\mathbf{M}_{sm}]\mathbf{x}_m = -\mathbf{K}_{ss}^{-1}[\mathbf{I} - \omega^2\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}]^{-1}[\mathbf{K}_{sm} - \omega^2\mathbf{M}_{sm}]\mathbf{x}_m$$

$$= -\mathbf{K}_{ss}^{-1}[\mathbf{I} + \omega^2\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1} + o(\omega^4)][\mathbf{K}_{sm} - \omega^2\mathbf{M}_{sm}]\mathbf{x}_m$$

$$= -\mathbf{K}_{ss}^{-1}[\mathbf{K}_{sm} + \omega^2(\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} - \mathbf{M}_{sm}) + o(\omega^4)]\mathbf{x}_m, \tag{6}$$

where $o(\omega^4)$ denotes an error of order ω^4. The object is to improve the natural frequency and mode shape estimation from the reduced model. The reduced model based on static reduction, to first order in ω^2, satisfies

$$\omega^2\mathbf{M}_R\mathbf{x}_m = \mathbf{K}_R\mathbf{x}_m, \qquad \text{or} \qquad \omega^2\mathbf{x}_m = \mathbf{M}_R^{-1}\mathbf{K}_R\mathbf{x}_m, \tag{7}$$

where $\mathbf{M}_R$ and $\mathbf{K}_R$ are the reduced mass and stiffness matrices obtained from static reduction, and ω and $\mathbf{x}_m$ are a natural frequency of the reduced model and the associated eigenvector at the master co-ordinates. Equation (6) becomes, upon ignoring the terms in ω^4 and higher powers

$$\mathbf{x}_s = [-\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} + \mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm} - \mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm})\mathbf{M}_R^{-1}\mathbf{K}_R]\mathbf{x}_m. \tag{8}$$

Equation (8) defines a transformation that generates the slave co-ordinates from the master co-ordinates. Although only strictly correct when the co-ordinate vector is a mode shape, it may be applied as a general transformation, $\mathbf{T}_{IRS}$, which may be conveniently written as [2]

$$\mathbf{T}_{IRS} = \mathbf{T}_s + \mathbf{SMT}_s\mathbf{M}_R^{-1}\mathbf{K}_R, \tag{9}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ss}^{-1} \end{bmatrix}.$$

The reduced mass and stiffness matrices obtained by using the IRS method are then

$$\mathbf{M}_{IRS} = \mathbf{T}_{IRS}^T\mathbf{MT}_{IRS}, \qquad \mathbf{K}_{IRS} = \mathbf{T}_{IRS}^T\mathbf{KT}_{IRS}. \tag{10}$$

Although equation (9) is a convenient form in which to express the IRS transformation, in practice it is inefficient to compute the transformation in this way. The transformation may be generated by using an expression similar to equation (8).

3. A DYNAMIC IRS PROCEDURE

The transformation in the standard IRS method may be viewed as a perturbation on the transformation from the static reduction method. A similar perturbation may be obtained for dynamic reduction. Static reduction is correct only at zero frequency. The transformation for the dynamic reduction method is correct at a given frequency Ω and the transformation equivalent to equation (3) is given by

$$\begin{Bmatrix} \mathbf{x}_m \\ \mathbf{x}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{I} \\ -(\mathbf{K}_{ss} - \Omega^2\mathbf{M}_{ss})^{-1}(\mathbf{K}_{sm} - \Omega^2\mathbf{M}_{sm}) \end{bmatrix}\{\mathbf{x}_m\} = \mathbf{T}_d\mathbf{x}_m. \tag{11}$$

A dynamic IRS method may be formulated in the same way as the standard IRS method except that the slave degrees of freedom are written as a power series in $(\omega^2 - \Omega^2)$. Thus, one obtains, in a manner similar to that for equation (6),

$$\begin{aligned} \mathbf{x}_s &= -[\mathbf{D}_{ss} - (\omega^2 - \Omega^2)\mathbf{M}_{ss}]^{-1}[\mathbf{D}_{sm} - (\omega^2 - \Omega^2)\mathbf{M}_{sm}]\mathbf{x}_m \\ &= -\mathbf{D}_{ss}^{-1}[\mathbf{I} - (\omega^2 - \Omega^2)\mathbf{M}_{ss}\mathbf{D}_{ss}^{-1}]^{-1}[\mathbf{D}_{sm} - (\omega^2 - \Omega^2)\mathbf{M}_{sm}]\mathbf{x}_m \\ &= -\mathbf{D}_{ss}^{-1}[\mathbf{I} + (\omega^2 - \Omega^2)\mathbf{M}_{ss}\mathbf{D}_{ss}^{-1} + o((\omega^2 - \Omega^2)^2)][\mathbf{D}_{sm} - (\omega^2 - \Omega^2)\mathbf{M}_{sm}]\mathbf{x}_m \\ &= -\mathbf{D}_{ss}^{-1}[\mathbf{D}_{sm} + (\omega^2 - \Omega^2)(\mathbf{M}_{ss}\mathbf{D}_{ss}^{-1}\mathbf{D}_{sm} - \mathbf{M}_{sm}) + o((\omega^2 - \Omega^2)^2]\mathbf{x}_m, \end{aligned} \tag{12}$$

where $\mathbf{D}_{ss} = \mathbf{K}_{ss} - \Omega^2\mathbf{M}_{ss}$ and $\mathbf{D}_{sm} = \mathbf{K}_{sm} - \Omega^2\mathbf{M}_{sm}$. The reduced model based on the dynamic reduction, to first order in $(\omega^2 - \Omega^2)$, satisfies

$$(\omega^2 - \Omega^2)\mathbf{M}_R\mathbf{x}_m = \mathbf{D}_R\mathbf{x}_m, \qquad \text{or} \qquad (\omega^2 - \Omega^2)\mathbf{x}_m = \mathbf{M}_R^{-1}\mathbf{D}_R\mathbf{x}_m, \tag{13}$$

where $\mathbf{D}_R = \mathbf{K}_R - \Omega^2\mathbf{M}_R$. Equation (13) applies only at the natural frequencies and mode shapes (compare with equation (7)). The dynamic IRS (DIRS) transformation is therefore

$$\mathbf{T}_{DIRS} = \mathbf{T}_d + \mathbf{S}_d\mathbf{M}\mathbf{T}_d\mathbf{M}_R^{-1}\mathbf{D}_R, \tag{14}$$

where

$$\mathbf{S}_d = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{ss}^{-1} \end{bmatrix}.$$

The reduced mass and stiffness matrices are given by expressions similar to equations (10).

4. ITERATED IRS TECHNIQUES

The transformation (9) relies on the reduced mass and stiffness matrices obtained from static reduction. Once the transformation has been computed, an improved estimate of these reduced matrices is available from equations (10). These improved estimates could be used in the definition of the IRS transformation, equation (9), to give a more acurate transformation. The transformation for the first iteration is given by equation (9), and for subsequent iterations is

$$\mathbf{T}_{IRS,i+1} = \mathbf{T}_s + \mathbf{S}\mathbf{M}\mathbf{T}_{IRS,i}\mathbf{M}_{IRS,i}^{-1}\mathbf{K}_{IRS,i}, \tag{15}$$

where the subscript i denotes the ith iteration. In equation (15) the transformation $\mathbf{T}_{IRS,i}$ is the current IRS transformation and $\mathbf{M}_{IRS,i}$ and $\mathbf{K}_{IRS,i}$ are the associated reduced mass and stiffness matrices given by equations (10). A new transformation, $\mathbf{T}_{IRS,i+1}$, is obtained which then becomes the current IRS transformation for the next iteration. Blair *et al*. [4] considered an iterated IRS technique, but the second term of equation (15) is changed to give

$$\mathbf{T}_{IRS,i+1} = \mathbf{T}_s + \mathbf{S}\mathbf{M}\mathbf{T}_s\mathbf{M}_{IRS,i}^{-1}\mathbf{K}_{IRS,i}. \tag{16}$$

Notice that the static transformation is retained in the second term, whereas the updated transformation is used in equation (15). The algorithm defined by (16) converges to yield reduced mass and stiffness matrices that do not reproduce the eigenvalues of the full system. The dynamic IRS method given in the previous section may easily be extended to an iterative scheme similar to (15).

A full proof that the iterated IRS method converges is extremely difficult and will not be given here. Convergence may be implied by considering the accuracy of the series expansion given by equation (6). The generation of the IRS method requires the approximation of the natural frequencies of the full system by using the reduced model, equation (7). At each iteration the reduced model will become more accurate and experience has shown that the iterated IRS method converges monotonically from the statically reduced model to the SEREP model (defined in reference [5]). The speed of convergence will depend on the choice of master degrees of freedom, and a poor choice will lead to a very slow rate of convergence. Choosing master degrees of freedom for which a low frequency mode is not observable leads to difficulties in SEREP reduction. In the iterated IRS method the modes that are not observable are simply not included in the reduced model which will converge to the lowest observable modes.

It will now be proved that when the iterative scheme given by equation (15) converges, the reduced model reproduces the lower observable eigenvalues and their associated reduced eigenvectors of the full system. On convergence the transformation is

$$\mathbf{T}_{IRS}=\begin{bmatrix}\mathbf{I}\\ \mathbf{t}_{IRS}\end{bmatrix}, \tag{17}$$

where

$$\mathbf{t}_{IRS}=-\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}+\mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm}+\mathbf{M}_{ss}\mathbf{t}_{IRS})\mathbf{M}_{IRS}^{-1}\mathbf{K}_{IRS}. \tag{18}$$

Assume that λ is an eigenvalue with its associated eigenvector ϕ_m of the converged, reduced system defined by $\mathbf{M}_{IRS}$ and $\mathbf{K}_{IRS}$. Let ϕ_s be the eigenvector at the slave co-ordinates estimated by using the converged transformation $\mathbf{T}_{IRS}$, so that

$$\phi=\begin{bmatrix}\phi_m\\ \phi_s\end{bmatrix}=\mathbf{T}_{IRS}\phi_m. \tag{19}$$

Then we have to show that λ is an eigenvalue of the full system with eigenvector ϕ. Since ϕ_m is an eigenvector of the reduced system

$$\mathbf{M}_{IRS}^{-1}\mathbf{K}_{IRS}\phi_m=\lambda\phi_m, \tag{20}$$

and equations (18), (19) and (20) may be combined to give

$$\phi_s=\mathbf{t}_{IRS}\phi_m=[-\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}+\lambda\mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm}+\mathbf{M}_{ss}\mathbf{t}_{IRS})]\phi_m, \tag{21}$$

or, by premultiplying by $\mathbf{K}_{ss}$ and using equation (19),

$$\mathbf{K}_{ss}\phi_s=-\mathbf{K}_{sm}\phi_m+\lambda\mathbf{M}_{sm}\phi_m+\lambda\mathbf{M}_{ss}\phi_s. \tag{22}$$

Rearranging gives

$$\lambda[\mathbf{M}_{sm}\quad \mathbf{M}_{ss}]\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}=[\mathbf{K}_{sm}\quad \mathbf{K}_{ss}]\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}. \tag{23}$$

Since λ is an eigenvalue of the reduced system with eigenvector ϕ_m,

$$\lambda\mathbf{T}_{IRS}^{\mathrm{T}}\mathbf{M}\mathbf{T}_{IRS}\phi_m=\lambda[\mathbf{I}\quad \mathbf{t}_{IRS}^{\mathrm{T}}]\mathbf{M}\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}$$

$$=[\mathbf{I}\quad \mathbf{t}_{IRS}^{\mathrm{T}}]\mathbf{K}\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}=\mathbf{T}_{IRS}^{\mathrm{T}}\mathbf{K}\mathbf{T}_{IRS}\phi_m. \tag{24}$$

Multiplying out expression (24) and using equation (23) or equivalently equation (22) produces

$$\lambda[\mathbf{M}_{mm} \quad \mathbf{M}_{ms}]\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}=[\mathbf{K}_{mm} \quad \mathbf{K}_{ms}]\begin{Bmatrix}\phi_m\\ \phi_s\end{Bmatrix}. \tag{25}$$

Combining equations (23) and (25) gives $\lambda\mathbf{M}\phi=\mathbf{K}\phi$ and shows that λ is an eigenvalue of the full system with associated eigenvalue ϕ.

One consequence of the convergence to the eigenvalues and eigenvectors of the full model is that, on convergence, the transformation obtained from the iterated IRS method is exactly the same as that produced by the SEREP method [5]. This convergence to the SEREP transformation is dependent on all the lower modes being observable, based on the choice of the master degrees of freedom. If the lower modes are not observable then the SEREP transformation is not defined. In this case a SEREP transformation may be obtained by choosing only the lowest observable modes, and then the iterated IRS transformation will converge to this redefined SEREP transformation.

5. NUMERICAL CONSIDERATIONS

We will now consider some practical issues relating to the implementation of the iterated IRS method. Of fundamental importance are the computational requirements of the method. In general, determining whether the iterated IRS requires more computation than the SEREP reduction is extremely difficult, particularly since eigenvector extraction and the iterated IRS method are both iterative procedures, and the number of iterations will depend on the properties of the full system and the choice of master degrees of freedom. For example, subspace iteration will require many iterations if the highest eigenvalue in the subspace is close to the lowest discarded eigenvalue. A poor choice of master degrees of freedom will require many iterations before the IRS method converges.

Although a full analysis of the computational requirements of the IRS method is difficult, a comparison can be made with static reduction. Static reduction is often implemented in commerical finite element codes, and has thus shown the computational benefits of reduction compared to a direct eigensolution. Even this limited comparison is difficult, due to the uncertainty in the implementation of the methods and the type of system modelled. To make the comparison possible, we will assume the mass and stiffness matrices are fully populated. Let n and N represent the number of master and slave degrees of freedom respectively. In general, $n \ll N$ and static reduction, performed one degree of freedom at a time, requires approximately $\frac{2}{3}N^3$ floating point multiplications. Calculating the transformation matrix involves the LU decomposition of $\mathbf{K}_{ss}$ and requires an extra $\frac{1}{3}N^3$ multiplications. The transformation must be calculated to determine the mode shapes at all the degrees of freedom. The iterated IRS method requires the computation of equation (15) at every iteration. The calculation involves only the lower part of the transformation given by

$$\mathbf{t}_{IRS,i+1}=-\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}+\mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm}+\mathbf{M}_{ss}\mathbf{t}_{IRS,i})\mathbf{M}_{IRS,i}^{-1}\mathbf{K}_{IRS,i} \tag{26}$$

where

$$\mathbf{T}_{IRS,i}=\begin{bmatrix}\mathbf{I}\\ \mathbf{t}_{IRS,i}\end{bmatrix}.$$

The first part of equation (26) is the static transformation. Before the iterated IRS method begins, the matrices $\mathbf{K}_{ss}^{-1}\mathbf{M}_{sm}$ and $\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}$ are calculated. If these matrices are calculated at the same time as the static reduction transformation, this requires, for $n \ll N$, approximately

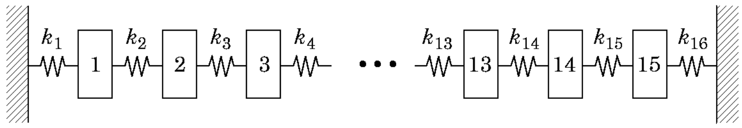

Figure 1. The discrete 15-degree-of-freedom example.

N^3 extra multiplications. For every iteration the reduced mass and stiffness matrices are calculated by using a total of $n(n+1)N^2$ multiplications. The other products in equation (26) require, for $n \ll N$, approximately nN^2 multiplications. Thus the iterated IRS method requires approximately N^3 extra multiplications initially and $n(n+2)N^2$ multiplications per iteration. The computational requirement per iteration is relatively small, and overall the iterated IRS will require an extra computational effort similar to that of the original static reduction.

6. A DISCRETE MASS–SPRING EXAMPLE

We will begin by illustrating the application of the IRS method to a simple mass–spring system. The system consists of 15 inertias joined together by 16 springs to form a chain, as shown in Figure 1. The inertias at the ends of the chain are connected to ground by springs. Each spring has a stiffness of 1 kN/m and each inertia has a mass of 1 kg. The full 15-degree-of-freedom model is reduced to one with five degrees of freedom by selecting the response at the 1st, 4th, 7th, 11th and 15th inertias. The natural frequencies for the models resulting from both the static reduced and the iterated IRS method, for ten iterations, are given in Table 1. The first iteration is the standard IRS method. The table shows that static reduction fails to reproduce any of the natural frequencies of the original analytical model. Furthermore, because the static reduction ignores inertia terms the higher frequency modes are the least accurate. The inertia terms are critical to the accurate estimation of the higher natural frequencies. When using the iterated IRS method, the lower frequencies converge most rapidly, although all frequencies will converge to the lower natural frequencies of the

Table 1

Convergence of natural frequencies for the 15-*dof example when using the iterated IRS procedure*

	Natural frequency (rad/s)				
Iterations	1	2	3	4	5
Static reduction	6·3227	13·2268	21·6133	29·8367	34·5388
1	6·1992	12·3437	18·5543	25·2854	31·2899
2	6·1991	12·3389	18·4018	24·6036	30·5049
3	6·1991	12·3386	18·3719	24·3860	30·2018
4	6·1991	12·3386	18·3637	24·2966	30·0573
5	6·1991	12·3386	18·3610	24·2542	29·9784
6	6·1991	12·3386	18·3600	24·2323	29·9309
7	6·1991	12·3386	18·3596	24·2204	29·9000
8	6·1991	12·3386	18·3594	24·2135	29·8788
9	6·1991	12·3386	18·3593	24·2095	29·8636
10	6·1991	12·3386	18·3593	24·2071	29·8523
Exact	6·1991	12·3386	18·3592	24·2030	29·8137

TABLE 2

Convergence of natural frequencies for the 15-*dof example* (*when using equation* (16) *in the iterative IRS procedure*)

Iterations	Natural frequency (rad/s) 1	2	3	4	5
Static reduction	6·3227	13·2268	21·6133	29·8367	34·5388
1	6·1992	12·3437	18·5543	25·2854	31·2899
2	6·1996	12·3740	18·9494	26·2979	31·8706
3	6·1996	12·3716	18·8841	25·9335	31·6153
4	6·1996	12·3722	18·8988	26·0674	31·8077
5	6·1996	12·3720	18·8951	26·0113	31·6263
6	6·1996	12·3721	18·8963	26·0350	31·8112
7	6·1996	12·3720	18·8958	26·0243	31·6226
8	6·1996	12·3721	18·8960	26·0290	31·8189
9	6·1996	12·3720	18·8959	26·0270	31·6179
10	6·1996	12·3721	18·8960	26·0276	31·8275
Exact	6·1991	12·3386	18·3592	24·2030	29·8137

full model after sufficient iterations. The transformation matrix converges to that obtained by the SEREP method [5].

In Table 2 is shown the effect of using the algorithm suggested by Blair *et al*. [4], equation (12), to update the transformation matrix. The natural frequencies of the reduced model converge, but not to the natural frequencies of the full model. The natural frequencies from the standard IRS method are closer to the frequencies of the full model than the fully converged frequencies when using this method.

In Table 3 is shown the effect of using a dynamic IRS iterative procedure, based on equation (16), with a central frequency of 12 rad/s ($\Omega = 12$ rad/s). As expected, the natural

TABLE 3

Convergence of natural frequencies for the 15-*dof example* (*when using the dynamic IRS iterative procedure*, $\Omega = 12$ *rad/s*)

Iterations	Natural frequency (rad/s) 1	2	3	4	5
Dynamic reduction	7·2094	12·3418	19·8337	28·4467	33·8911
1	6·2029	12·3386	18·3947	24·7950	30·9329
2	6·1993	12·3386	18·3646	24·3947	30·3046
3	6·1992	12·3386	18·3604	24·2808	30·0798
4	6·1991	12·3386	18·3596	24·2384	29·9769
5	6·1991	12·3386	18·3593	24·2203	29·9214
6	6·1991	12·3386	18·3593	24·2119	29·8880
7	6·1991	12·3386	18·3592	24·2077	29·8663
8	6·1991	12·3386	18·3592	24·2056	29·8516
9	6·1991	12·3386	18·3592	24·2044	29·8414
10	6·1991	12·3386	18·3592	24·2038	29·8341
Exact	6·1991	12·3386	18·3592	24·2030	29·8137

TABLE 4

Convergence of natural frequencies for the 15-*dof example when using the iterated IRS procedure where one mode is not observable*

Iterations	Natural frequency (rad/s) 1	2	3	4	5
Static reduction	3·4409	10·5616	19·0172	26·0810	34·5282
1	3·4240	10·2327	16·9988	23·8133	31·2802
2	3·4240	10·2320	16·9303	23·5304	30·4696
3	3·4240	10·2320	16·9218	23·4549	30·1272
4	3·4240	10·2320	16·9204	23·4290	29·9463
5	3·4240	10·2320	16·9201	23·4186	29·8394
6	3·4240	10·2320	16·9200	23·4140	29·7721
7	3·4240	10·2320	16·9200	23·4119	29·7280
8	3·4240	10·2320	16·9200	23·4108	29·6981
9	3·4240	10·2320	16·9200	23·4102	29·6776
10	3·4240	10·2320	16·9200	23·4100	29·6631
Exact	3·4240	10·0000	10·2320	16·9200	23·4096

frequency closest to 12 rads/s, that is the second frequency, converges most quickly. More iterations would confirm that the natural frequencies do indeed converge to those of the full model.

The discrete example will now be used to demonstrate the effect of an unobservable mode in the frequency range of interest. Suppose that the stiffnesses k_{15} and k_{16} in the system shown in Figure 1 are set to $k_{15}=0$ and $k_{16}=100$ Nm^{-1}. Thus mass number 15 is decoupled from the other 14 masses. Furthermore, the stiffness k_{16} is chosen so that the local mode involving mass 15 is the second natural frequency of the full system. Suppose that we choose the response of the 1st, 4th, 7th, 11th and 14th masses as the master degrees of freedom. Now the local mode involving mass 15 is not observable. The effect of applying the iterated IRS method to this system is shown in Table 4. Notice that the method has converged to natural frequencies of the full system, but the second mode has been missed out since it is not observable.

7. EXAMPLES OF CONTINUOUS STRUCTURES

We now illustrate the application of the iterative IRS method to a discrete model of a continuous structure. The structure considered is a plate with a slot on one side and clamped along two other sides, as shown in Figure 2. The full model has 90 degrees of freedom and

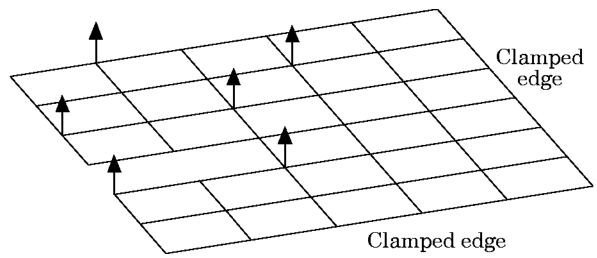

Figure 2. The finite element model of the slotted plate, showing the reduced co-ordinates.

TABLE 5
Convergence of natural frequencies for the plate example when using the iterated IRS procedure

Iteration	Natural frequency (rad/s) 1	2	3	4	5	6
Static reduction	30·5886	108·257	147·269	225·203	328·732	441·383
1	30·5162	104·745	141·253	208·938	273·084	368·303
2	30·5162	104·740	141·233	208·587	265·959	361·732
3	30·5162	104·740	141·232	208·538	263·929	359·185
4	30·5162	104·740	141·232	208·527	263·193	357·846
5	30·5162	104·740	141·232	208·524	262·885	357·009
6	30·5162	104·740	141·232	208·523	262·744	356·426
7	30·5162	104·740	141·232	208·522	262·677	355·992
8	30·5162	104·740	141·232	208·522	262·642	355·654
9	30·5162	104·740	141·232	208·522	262·624	355·386
10	30·5162	104·740	141·232	208·522	262·615	355·169
Exact	30·5162	104·740	141·232	208·522	262·603	354·107

is reduced to the six degrees of freedom shown in Figure 2. This choice of co-ordinates represents the best selection of co-ordinates for Guyan reduction on the basis of the relative importance of the diagonal stiffness and inertia terms [6, 7]. The result of using the iterative IRS method on this example is shown in Table 5. The convergence of the natural frequencies of the reduced model to those of the full model is clear. As expected, the lower modes converge more quickly than the higher modes. The result of applying the iterative dynamic IRS method based on a frequency of 150 rad/s is shown in Table 6. The natural frequencies close to 150 rad/s converge quicker than those frequencies further away.

Our third example concerns a two dimensional frame shown in Figure 3 and modelled with 34 nodes. Each node has three degrees of freedom and two of the nodes are grounded,

TABLE 6
Convergence of natural frequencies for the plate example (*when using the dynamic IRS iterative procedure*, $\Omega = 150$ *rad/s*)

Iteration	Natural frequency (rads/s) 1	2	3	4	5	6
Dynamic reduction	58·5194	109·257	141·350	213·130	305·242	423·897
1	30·5561	104·750	141·232	208·566	266·898	364·963
2	30·5173	104·740	141·232	208·526	263·669	359·799
3	30·5162	104·740	141·232	208·523	262·949	357·851
4	30·5162	104·740	141·232	208·522	262·730	356·812
5	30·5162	104·740	141·232	208·522	262·653	356·145
6	30·5162	104·740	141·232	208·522	262·624	355·674
7	30·5162	104·740	141·232	208·522	262·612	355·326
8	30·5162	104·740	141·232	208·522	262·607	355·062
9	30·5162	104·740	141·232	208·522	262·605	354·859
10	30·5162	104·740	141·232	208·522	262·604	354·702
Exact	30·5162	104·740	141·232	208·522	262·603	354·107

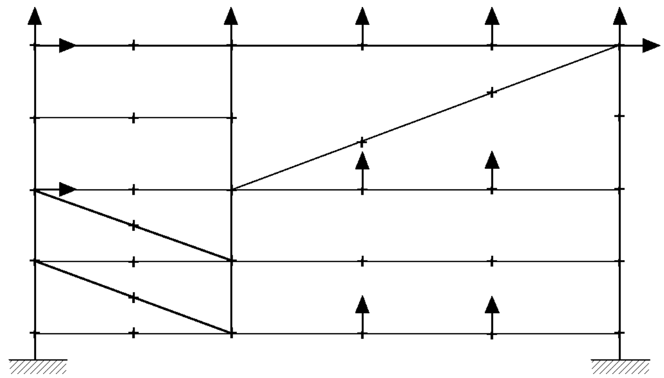

Figure 3. The frame example showing the node positions (+) and the reduced co-ordinates (arrows).

giving 96 degrees of freedom in the full model. This model is reduced to the 12 degrees of freedom shown in Figure 3. This co-ordinate selection is not the optimum set for the application of static reduction [7] but is satisfactory for our purposes. Applying the iterated IRS method to this example gives the results shown in Table 7. Only the first six frequencies are shown. The convergence is satisfactory, although not particularly rapid. A better choice of the reduced co-ordinates would help to speed the convergence. Our first two examples demonstrated the iterated, dynamic IRS method when the centre frequency, Ω, is relatively small. What happens if a large centre frequency is used? In this example the 12th natural frequency of the full model is at approximately 4455 rad/s. Suppose that we tried a centre frequency of 3000 rad/s. The iterated DIRS method converges to the higher natural frequencies of the full system and the first three frequencies are missed. One major problem is the appearance of spurious natural frequencies in the reduced model. These spurious frequencies are easily recognized as they are not constant. In the current example, the effect is demonstrated in Figure 4. On a diagram such as this it is easy to pick out the true natural frequencies, which accurately reproduce those of the full model. Notice that the natural frequencies closest to $\Omega = 3000$ rad/s converge the fastest.

TABLE 7

Convergence of natural frequencies for the frame example

	Natural frequency (rad/s)					
Iterations	1	2	3	4	5	6
Static reduction	471·776	1536·00	1712·52	1914·52	2041·44	2702·54
1	464·687	1170·34	1646·01	1690·56	1903·97	1922·29
2	464·679	1145·59	1638·26	1671·79	1886·98	1907·10
3	464·676	1139·89	1635·11	1667·04	1880·62	1905·43
4	464·675	1136·77	1633·07	1664·71	1877·08	1904·78
5	464·674	1134·78	1631·62	1663·35	1874·79	1904·43
6	464·674	1133·41	1630·54	1662·46	1873·20	1904·21
7	464·674	1132·41	1629·72	1661·85	1872·03	1904·06
8	464·673	1131·66	1629·08	1661·40	1871·14	1903·95
9	464·673	1131·08	1628·57	1661·07	1870·44	1903·87
10	464·673	1130·61	1628·15	1660·81	1869·88	1903·80
Exact	464·672	1127·42	1624·88	1659·24	1865·83	1903·45

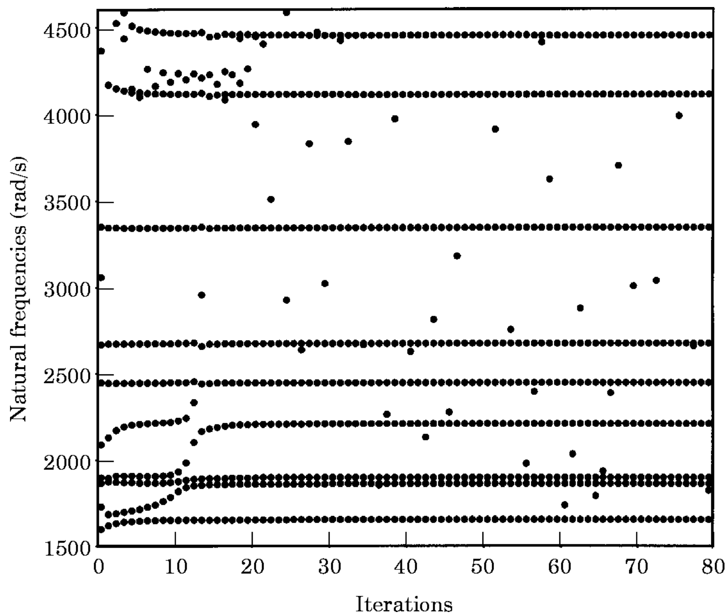

Figure 4. The convergence of the natural frequencies when using the dynamic IRS method with a large centre frequency $\Omega = 3000$ rad/s.

8. CONCLUSIONS

An iterative method based on the Improved Reduced System (IRS) algorithm has been presented. The natural frequencies of the reduced model converge to the lower frequencies of the full model, with those of lower frequency converging more quickly. A dynamic IRS method, based on dynamic rather than static reduction has been derived and tested. The iterative version of this method has also been derived. For a small centre frequency, the natural frequencies of the reduced model converge to the lower frequencies of the full model. The natural frequency of the reduced model closest to the chosen centre frequency for the reduction converges quickest. If the centre frequency is large then the iterated dynamic IRS method misses the lower frequencies of the full model. Spurious frequencies are introduced, although they are easily identified as they are not constant from one iteration to the next.

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