ON POLE–ZERO PLACEMENT BY UNIT-RANK MODIFICATION

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The dynamic behaviour of a general linear system is characterised by the poles (resonances) and the zeros (antiresonances). This paper deals with the general class of dynamical systems that can be described by a second-order ordinary differential equation. For spatially discretised models the poles are given by the roots of the characteristic polynomial of the system matrix whilst the zeros correspond to the roots of the polynomials related to the elements of the adjugate of the system matrix. The question of how the poles and zeros can be assigned by a structural modification is important in many areas of application, for example in control, vibration absorption and model updating. In this paper we derive the basic equation of pole–zero placement by unit-rank modification. For simple real matrix pencils we simplify the basic equation which enables an analytical solution of the transformed vector of the modification. These results are related to the original unit-rank modification by a system of linear equations. Finally, we will give insight into solution methods of the original problem, and suggest a method which is based on the incorporation of additional model structure information.

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1. INTRODUCTION

The well-known problem of pole-assignment in active control has useful applications in structural dynamics, especially when a modification to the natural frequencies of a mechanical system is contemplated. This problem falls into the area generally known as Structural Modification. The term ‘eigen-structure assignment’ [1–5] refers to the assignment of poles and residues, or equivalently eigenvalues and eigenvectors. The assignment of zeros as well as poles is related to eigen-structure assignment, although assigning zeros directly is less popular in control applications. Zeros, or antiresonances, are more important in vibration analysis where the force is often restricted to a single frequency, for example when the excitation arises from a rotating machine. Schrader and Sain [6] surveyed research into system zeros, including zero placement strategies. Typical zero assignment methods in active control problems are given by Karcanias and
Kouvaritakis [7], Murdoch [8], Cameron and Zolghadri-Jahromi [9], and Tu and Lin [10]. Ram [11] considered the application of pole–zero assignment to vibrating systems and Schulz and Inman [12] gave an example of using eigenstructure assignment for vibration suppression. Of course, the assignment of poles and zeros by active means requires the selection of control gains to bring about the desired modification. Its implementation calls for an external power source and the use of actuators to supply the necessary forces. Cannon’s book [13] is unusual in the control literature in that the assignment of poles by passive stiffness and mass modifications is discussed.

There are of course practical limitations to both active and passive modification of structures, but active methods generally offer greater flexibility because modifications can be considered which would be impossible to achieve passively. However, passive methods are often used in aircraft as secondary systems which come into use should the primary, active, system fail. The best known passive modification is the dynamic vibration absorber, a device with a history of almost 100 years [14] and still regularly used in aeroplanes, helicopters and road vehicles. A vibration absorber is able to assign a single antiresonance whilst changing all the other antiresonances and natural frequencies in a predictable way. Mottershead [15] generalised the theory of vibration absorption by using measured receptances to assign an antiresonance by a diadic modification. The method described in [15] differs from the standard ‘control’ approach in that there is no need for a mathematical model of the system. In the present article the authors revisit the problem of assigning the eigenvalues (poles and zeros) of a matrix system known to represent the dynamic behaviour of a mechanical system. The general case of a unit-rank modification is considered. The practical implementation of a single unit-rank modification passively is limited, but it may be possible to physically construct modifications of higher rank by the accumulation of a series of unit-rank adjustments. This approach was suggested by Akgun et al. [16] in the related problem of structural reanalysis using the Sherman–Morrison–Woodbury formulas.

The approach adopted in this paper is to consider exact solutions to the pole–zero placement problem. The alternative is a direct optimisation based on a pre-selected set of physical parameters of the system. This direct optimisation approach also allows inequality constraints on the poles and zeros, and also on the parameter values, to be introduced. Although optimisation is more general, any insight into the range of solutions to the placement problem is lost.

The paper is organised in five sections. The first section gives some background on poles and zeros. In Section 2, we derive the basic equation for the placement of roots by a unit-rank modification $xy^\top$ in which the $x$ and $y$ are linearly independent. The necessary and sufficient conditions for the existence of a solution are formulated and represent a non-linear problem in $x$. The following two sections are concerned with a simplification of the basic equation with the goal of obtaining an overview of all the solutions. In Section 4, a solution method for simple matrix pencils is presented which leads to analytical solutions for the transformed vectors $x,y$. In Section 5 this method is extended to the placement of zeros of defective minors of simple systems. It is shown that the problem of the back-transformation to the original vectors $x,y$ is equivalent to the solution of two linear problems. The coupling of both problems also leads to a non-linear problem which is in general difficult to solve. Hence an alternative method is suggested which is based on the fact that, in general, the unit-rank modification is not arbitrary. Often the vectors $x,y$ are restricted to certain substructures which are related to realistic system modifications, for instance the placement of additional lumped masses or dashpots. This issue is addressed.
in Section 6. The sections are accompanied by simple numerical examples where all the numerical values are rounded unless given as rational numbers. A summary of the nomenclature is given in the Appendix.

2. BACKGROUND

We focus our investigation on spatially discretised linear dynamic models with \( n \) degrees-of-freedom (dof) that may be described by

\[
A(s)u(s) = f(s), \quad s \in \mathbb{C}
\]

where \( u(s), f(s) \in \mathbb{C}^n \) are the output and input vectors, respectively, and

\[
A(s) := \sum_{\ell=0}^d s^\ell A_\ell \in \mathbb{C}^{n \times n}, \quad A_\ell \in \mathbb{R}^{n \times n}
\]

is a general representation of the system matrix by a matrix pencil of degree \( d \in \mathbb{N} \) [17]. Although some of the results presented in this paper hold for the general case \( d \in \mathbb{N} \) we will focus our attention on the case \( d \leq 2 \) where for \( d = 2 \), \( A_0, A_1, A_2 \in \mathbb{R}^{n \times n} \) represent the matrices \( K, D \) and \( M \) of stiffness, damping and inertia, respectively. Alternatively we may describe a general viscously damped system by a matrix pencil of degree 1, either by

\[
A_1 = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}
\]

or by

\[
A_1 = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & K \\ K & D \end{bmatrix}
\]

We will choose description (3) for a unit-rank modification of the stiffness or damping matrices whilst description (4) enables a unit-rank modification of the damping or mass matrices.

For given input \( f(s) \) the output can be calculated as

\[
u(s) = A(s)^{-1}f(s) = \frac{1}{\det(A(s))}A(s)^{ad}f(s)
\]

which implies that the dynamics of the system is governed by the roots of the polynomials

\[
p_0(s) := \det(A(s))
\]

\[
p_{0\ell}(s) := e_i^T A(s)^{ad} e_k = (-1)^{i+k}\det(A^{(\ell|\ell)}(s)).
\]

If \( A_d \) is non-singular the degrees of the polynomials are

\[
\deg(p_0) = dn
\]

\[
\deg(p_{0\ell}) \leq d(n-1).
\]

The last inequality is due to a potential rank deficiency of a minor of \( A_d \). We will address this issue later. Common roots of \( p_0 \) and \( p_{0\ell} \) factor out in equation (5) and hence will not contribute to the dynamic behaviour. This subject of pole–zero cancellation has been studied elsewhere (for instance [18] or [19]). It is clear from equation (5) that the roots of \( p_0 \) are related to the maximum dynamic response (resonances) whilst the roots of \( p_{0\ell} \) determines the minimum dynamic response (antiresonances, nodes). Of course, since \( A_\ell \) (and hence \( A^{(\ell|\ell)}_\ell \)) are real-valued, for every pole (or zero) \( \lambda \in \mathbb{C} \) the conjugate \( \lambda^* \) is also a
pole (or zero). Before we formulate the problems analysed in this paper we introduce the following definitions.

**Definition 1.** We call the determinant of $S^T A(s) S_k = A^{(i,k)}(s) \in \mathbb{R}^{(n-1)\times(n-1)}$ the minor of the system $A(s)$ with respect to the index pair $(i, k)$, denoted briefly as minor $(i, k)$. Principal minors are those for which $i = k$.

**Definition 2.** If $\lambda$ is a root of $p_0(s)$ then the smallest number $n_A = n_A(\lambda) \in \mathbb{N}$ with

$$
\frac{\partial^n p_0(s)}{\partial s^n} \bigg|_{s=\lambda} = 0
$$

is called the algebraic multiplicity of $\lambda$, and the natural number

$$
n_G(\lambda) = n_G := n - \text{rank}(A(\lambda))
$$

is called the geometric multiplicity of $\lambda$.

**Definition 3.** A root $\lambda$ is called defective if $n_G < n_A$. We call the system described by the pencil $A(s)$ defective if $p_0$ has a defective root; otherwise the system is termed simple (or not defective) [17, p. 17].

In this paper, we will analyse the following basic problem of a unit-rank modification of a general (defective or simple) matrix pencil $A(s)$ of degree $d$:

Given $m$ disjoint complex numbers $\lambda_j, j \in \tilde{m}$, and $\ell \in \tilde{d}_0$ determine all vectors $x, y \in \mathbb{R}^n$ such that $\det(A(\lambda_j) + \lambda_j^\ell x y^\top) = 0$ for all $i \in \tilde{m}$.

Note that in the light of Definition 2, the above problem definition does not imply the placement of repeated roots. This issue has been addressed elsewhere [20] and is beyond the scope of this paper.

### 3. THE BASIC EQUATION

In order to establish an equation for the placement of roots (poles or zeros) some care is needed. There are two reasons for this: (1) Even in the case of non-singular matrices $A_\ell$ the minors of the matrix pencil may be singular. (2) If the $A_\ell$ are symmetric, their non-principal minors are not symmetric. To clarify the situation we consider the following simple example.

**Example 1.** Consider the following matrix pencil of degree 1:

$$
A_1 = I_3, \quad A_0 = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 2
\end{bmatrix}
$$

then the characteristic polynomial $p_0(s) = \det(sA_1 + A_0)$ has the roots $-1, -2$ and $-4$. Looking at the minor $(1, 3)$ we have

$$
A_1^{(1,3)} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
$$

$$
A_0^{(1,3)} = \begin{bmatrix}
-1 & 3 \\
0 & -1
\end{bmatrix}
$$

and hence the polynomial

$$
p_{013}(s) = \det(sA_1^{(1,3)} + A_0^{(1,3)}) = 1
$$
is constant. Now consider the minor $(1, 2)$ which leads to

\[
A_{1}^{(1|2)} = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
A_{0}^{(1|2)} = \begin{bmatrix}
-1 & -1 \\
0 & 2
\end{bmatrix}
\]

and then

\[
p_{012} = -\det(sA_{1}^{(1|2)} + A_{0}^{(1|2)}) = (s + 2).
\]

On the other hand, the minor $(1, 1)$ yields

\[
A_{1}^{(1|1)} = I_2
\]

\[
A_{0}^{(1|1)} = \begin{bmatrix}
3 & -1 \\
-1 & 2
\end{bmatrix}
\]

and hence

\[
p_{011}(s) = \det(sA_{1}^{(1|1)} + A_{0}^{(1|1)}) = s^2 + 5s + 5
\]

which has the two roots \( \lambda_1 = (-5 + \sqrt{5})/2 \) and \( \lambda_2 = -(5 + \sqrt{5})/2 \).

The degrees of the polynomials \( p_0 \) and \( p_{0ik} \) of a general matrix pencil of degree \( d \) depend on the structure and on the ranks of the \( A_\ell \). If \( A_d \) is non-singular then the degree of \( p_0 \) is \( dn \) and the roots correspond to the eigenvalues. However the above example demonstrates that for \( i \neq k \) the degree of \( p_{0ik} \) has the upper bound \( d(n - 1) \) which implies that for a matrix pencil with singular matrices the roots are not eigenvalues. To evaluate the exact degree of the polynomials in general is a complicated task.

For the following investigation we assume that the number of roots to be placed does not exceed the degree of the associated polynomial. Of course, since the \( A_\ell \) are real-valued, the number of complex roots to be placed can be reduced because for every complex root its conjugate is also a root.

We will now derive the basic equation of root placement by unit-rank modification. Let \( A(s) \in \mathbb{C}^{n \times n} \) denote an arbitrary matrix pencil of degree \( d \in \mathbb{N} \). Given disjoint complex numbers \( \lambda_i, i \in \tilde{m} \), we search for all vectors \( x, y \in \mathbb{R}^n \) such that for a fixed \( \ell \in \tilde{d}_0 \) the polynomial \( p(s) := \det(A(s) + s' xy^\top) \) has the \( m \) roots \( \lambda_i \). Using a well-known formula (see for instance [21, p. 65]),

\[
0 = \det(A(\lambda_i) + \lambda_i \ell' xy^\top)
\]

\[
= \det(A(\lambda_i)) + \lambda_i \ell' x^\top (A(\lambda_i))^{ad - \top} y
\]

and hence

\[
-\det(A(\lambda_i)) = x^\top \left[ \lambda_i \ell' (A(\lambda_i))^{ad - \top} \right] y.
\]

This equation can be extended for all \( i \in \tilde{m} \) to give

\[
g = T(x)y
\]
where the $i$th element of the vector $g \in \mathbb{C}^n$ is $-g_i$ and where

$$T(x) := \begin{bmatrix} x^T H_1 \\ \vdots \\ x^T H_m \end{bmatrix} \in \mathbb{C}^{m \times n}. \quad (15)$$

Note that $g = 0$ only if all $\lambda_i, i \in \bar{m}$, are already roots of $p_0 = \det(A(x))$. If none of the roots to be placed are zero and they are not roots of the unmodified system then all $H_i$ are non-singular.

Since we are looking for real-valued solutions, equation (14) can be written in real-valued form by doubling the order, i.e.

$$g_R = T_R(x)y \quad (16)$$

with

$$g_R := \begin{bmatrix} \Re\{g_i\} \\ \Im\{g_i\} \end{bmatrix} \in \mathbb{R}^{2m} \quad (17)$$

and where

$$T_R(x) := \begin{bmatrix} \Re\{T(x)\} \\ \Im\{T(x)\} \end{bmatrix} \in \mathbb{R}^{2m \times n}. \quad (18)$$

For fixed $x$ equation (16) states that the vector $g_R$ is contained in the subspace spanned by the columns of $T_R(x)$. This statement depends only on $x$ and is independent of $y$. The following proposition describes the set of all such $x$.

**Proposition 1.** Let $g_R \neq 0$ defined by equations (13) and (17), and let $T_R(x)$ be defined by equations (13),(15) and (18). Then $g_R \in \text{span}(T_R(x))$ if and only if

$$N_{T_R(x)}g_R = 0 \quad (19)$$

where $N_{T_R(x)} := I_{2m} - T_R(x)T_R(x)^+$ is the projector into the orthogonal complement of $\text{span}(T_R(x))$ and where the superscript $+$ denotes the Moore–Penrose inverse.

**Proof.** For fixed $x \neq 0$ the solution of $\min_{y \in \mathbb{R}^n} \|g - T(x)y\|^2$ is

$$\hat{y} := T_R(x)^+g_R \quad (20)$$

and this is a solution of equation (16) if and only if the equation error $g_R - T_R(x)\hat{y} = g_R - T_R(x)T_R(x)^+g_R = N_{T_R(x)}g_R$ is zero which means $g_R \in \text{span}(T_R(x))$. \qed

Although this proposition represents a condition for the existence of a solution of equation (16) the evaluation of condition (19) is a non-linear problem which is difficult to solve numerically because the projector $N_{T_R(x)}$ depends non-linearly on $x$. Whilst the type of non-linearity of this problem involves an inversion via the Moore–Penrose inverse the original problem, equation (16), is linear with respect to each unknown $x$ and $y$. The structure of the original problem can be revealed by rewriting equation (13) as

$$g_i = f_i^\top (x \otimes y) \Rightarrow g = F^\top (x \otimes y) \quad (21)$$
where \( \otimes \) denotes the Kronecker product and where \( F := [f_1, \ldots, f_m] \in \mathbb{C}^{m \times n^2} \) with the vector \( f_i \in \mathbb{C}^n \) consisting of the series of columns of \( H_i^T \). To ensure a real-valued solution we write (21) as

\[
g_R = F_R^T (x \otimes y)
\]

with \( F_R := [\text{Re}\{F\}, \text{Im}\{F\}] \). This equation is equivalent to equation (16).

In the next section, we introduce a method which enables an analytical solution of the transformed vectors \( x \) and \( y \). These transformed vectors are related to the original vectors by a system of linear equations which can be either over- or underdetermined. We defer the discussion of solving the non-linear problem (22) to a later section.

4. A PLACEMENT THEOREM FOR SIMPLE MATRIX PENCILS

Without loss of generality, we may assume that all matrices \( A_\ell \) of the matrix pencil with degree \( d \leq 2 \) can be diagonalised simultaneously. For systems with non-proportional damping this implies that the state space representation, equations (3) or (4), is used. Then there exist non-singular matrices \( Z, Q \in \mathbb{C}^{n \times n} \) such that \( A_\ell = Z \Gamma_\ell Q \), with the diagonal matrices \( \Gamma_\ell = \text{diag}(\gamma_{k\ell}) \). Without loss of generality we may assume that \( \det Z = \det Q = 1 \). Thus we have

\[
A(\lambda_i) = \sum_{\ell=0}^d \lambda_i^\ell A_\ell = Z \left( \sum_{\ell=0}^d \lambda_i^\ell \Gamma_\ell \right) Q
\]

and by definition (13)

\[
H_i = \lambda_i^\ell \Gamma_i A(\lambda_i)^{ad \top} = \lambda_i^\ell \Gamma_i Z^{-\top} \Gamma_i^{ad} Q^{-\top}
\]

Inserting this result into equation (14) we find

\[
g = T(x)y = T_0(Z^{-1}x)Q^{-\top}y = T_0(x_0)y_0
\]

where the transformed vectors are

\[
x_0 = Z^{-1}x, \quad y_0 = Q^{-\top}y
\]

and with the definition

\[
T_0(x_0) := \begin{bmatrix}
\lambda_i^\ell x_0^{\top} y_1^{ad} \\
\vdots \\
\lambda_i^m x_0^{\top} y_m^{ad}
\end{bmatrix}.
\]

Since \( \Gamma_i \) is diagonal its adjugate is also diagonal [22], its \( k \)th diagonal element is

\[
(\Gamma_i^{ad})_{kk} = \prod_{\ell \neq k}^d \sum_{j=0}^d \lambda_i^j \gamma_{j\ell} =: c_{ki}
\]

and hence

\[
\Gamma_i^{ad} = \text{diag}(c_i)
\]
which is a matrix with the vector \( c_i = [c_{i1}, c_{i2}, \ldots, c_{in}]^T \in \mathbb{C}^n \) on its diagonal. We can now rewrite equation (27) as

\[
T_0(x_0) = \begin{bmatrix}
\hat{\lambda}_1 c_1^T \text{diag}(x_0) \\
\vdots \\
\hat{\lambda}_m c_m^T \text{diag}(x_0)
\end{bmatrix}
= \Lambda^T C^\top \text{diag}(x_0),
\]

(30)

where \( \Lambda = \text{diag}_{i \in \mathbb{N}}(\hat{\lambda}_i) \) and where the \( i \)th column of \( C \in \mathbb{C}^{n \times m} \) is \( c_i \in \mathbb{C}^n \). The basic equation (14) now reads as

\[
g = \Lambda^T C^\top \text{diag}(x_0)y_0.
\]

(31)

We just have derived the result of a placement theorem for simple matrix pencils.

**Theorem 1.** Let the system described by \( \sum_{j=0}^d s^j A_j \) be simple. For fixed \( \ell \in \mathbb{Z}_0 \) the disjoint complex numbers \( \hat{\lambda}_i, i \in \mathbb{N}, \) are roots of \( p(s) := \det(\sum_{j=0}^d s^j A_j + s\ell xy^\top) \) if equation (31) has a solution \( x_0, y_0 \) such that \( ZX_0, Q^\top y_0 \in \mathbb{R}^n \).

Note that in the light of equations (3) and (4), Theorem 1 can be applied to a wide class of simple systems. A corresponding example will be presented later in this paper.

We will now explore methods of solving equation (31). If \( A \) is non-singular and if \( C \in \mathbb{C}^{n \times m} \) has full rank \( m \leq n \) then equation (31) has the general solution

\[
\text{diag}(x_0)y_0 = C^+ \Lambda^{-\ell} g + NC\eta
\]

(32)

for arbitrary \( \eta \in \mathbb{C}^{n-m} \), where the columns of \( NC \in \mathbb{C}^{n \times (n-m)} \) span the kernel of \( C^\top \). \( \eta \) may be chosen to satisfy the constraints that \( x \) and \( y \) should be real. The \( i \)th of the \( n \) equations represented by equation (32) reads

\[
x_{0i}y_{0i} = h_i + e_i^\top NC\eta.
\]

(33)

In principle the problem is solved. In the general case we can express \( y_{0i} \) in terms of \( x_{0i} \neq 0 \) and \( \eta \)

\[
y_{0i} = \frac{h_i + e_i^\top NC\eta}{x_{0i}}
\]

(34)

and if \( x_{0i} = 0 \) for some \( i \in \mathbb{N} \) then \( \eta \) has to satisfy

\[
e_i^\top NC\eta = -h_i.
\]

(35)

In the special case (see Example 3) of a symmetric modification \( y = xz \), \( \neq z \in \mathbb{R} \), of a symmetric matrix pencil we have \( Z = Q^\top \) and hence \( x_0 = xyz_0 \). Equation (33) now reads as

\[
x_{0i}^2 = \frac{h_i + e_i^\top NC\eta}{z^2} =: u_i^\top p
\]

(36)

where \( p_1 := 1/z, [p_2, \ldots, p_{n-m+1}] = \eta^\top/z, \) and where the vector \( u_i \) is defined elementwise by \( u_{1i} := h_i, (u_{2i}, \ldots, u_{(n-m+1)i}) := e_i^\top NC \). Equation (36) has the two solutions

\[
x_0(p) := \pm u_i^\top p
\]

(37)
for every $p \in \mathbb{R}^+ \times \mathbb{C}^{n-m}$ which means that the first component of $p$ has to be real and positive. However in both cases the remaining problem is the back-transformation equation (26) because in general $Q, Z, h, N_C$ and $\eta$ are complex-valued. We focus on the case of a symmetric modification of a symmetric matrix pencil which is important for a wide range of dynamic systems. In this case the back-transformation of equation (37) yields

$$x(p) := Zx_0(p)$$

which is real-valued if

$$\text{Re}\{Z\}\text{Im}\{x_0(p)\} + \text{Im}\{Z\}\text{Re}\{x_0(p)\} = 0$$

or equivalently

$$Gv(q) = 0$$

where $v(q)^\top := (\text{Im}\{x_0(p)\}^\top, \text{Re}\{x_0(p)\}^\top)$ with $q^\top := (1, \text{Re}\{\eta\}^\top, \text{Im}\{\eta\}^\top)/x$ and where $G := [\text{Re}\{Z\}, \text{Im}\{Z\}]$. Although the solution of equation (40) for $q \in \mathbb{R}^{2(n-m)+1}$ is a non-linear problem the type of non-linearity involved is easier to handle numerically than the non-linear problem given by equation (19). The situation becomes much easier if the quantities involved are real-valued. We will now study such an example.

**Example 2.** Let $A_1 = I_3$ and

$$A_0 = \frac{1}{12} \begin{bmatrix} 44 & -8 & -8 \\ 13 & 20 & -7 \\ 4 & 8 & 8 \end{bmatrix}$$

then the roots of $\det(sA_1 + A_0)$ are $(-1, -2, -3)$. The absolute values of the nine elements of the transfer matrix $(A_1 + sA_0)^{-1}$ are shown in Fig. 1 for $s \in [-4, 2]$. Clearly the system has peaks at the poles $(-1, -2, -3)$. Now suppose we want to place the poles at $(\lambda_1, \lambda_2, \lambda_3) = (-0.5, -1.5, -2.5)$ then we have $q^\top = [-15 - 3 3]/8$ and for the case $\ell = 0$ we find (see equation (13))

$$H_1 = \frac{1}{24} \begin{bmatrix} 14 & -9 & 8 \\ -8 & 18 & -56 \\ 28 & 27 & 106 \end{bmatrix}$$

$$H_2 = \frac{1}{24} \begin{bmatrix} 6 & 17 & 16 \\ -24 & -38 & -40 \\ 12 & 13 & 26 \end{bmatrix}$$

$$H_3 = \frac{1}{24} \begin{bmatrix} 46 & 43 & 24 \\ -40 & -46 & -24 \\ -4 & -1 & -6 \end{bmatrix}.$$ 

Now suppose $x = e_3$ then, according to equation (14), we have

$$T(e_3) = \frac{1}{24} \begin{bmatrix} 28 & 27 & 106 \\ 12 & 13 & 26 \\ -4 & -1 & -6 \end{bmatrix}$$

which is non-singular. Hence

$$y(e_3) = T(e_3)^{-1}g = [-93 12 30]^\top/20.$$
For this particular unit-rank modification, $xy^\top = e_3y(e_3)^\top$, the modified matrix is

$$A_0 + xy^\top = \frac{1}{60} \begin{bmatrix} 220 & -40 & -40 \\ 65 & 100 & -35 \\ 299 & 4 & -50 \end{bmatrix}$$
and the modified system \( sA_1 + A_0 + xy^\top \) does indeed have the desired poles at \((-0.5, -1.5, -2.5)\). To obtain an overview of all the solutions we note that the matrices \( A_1 \) and \( A_0 \) are simultaneously diagonalisable with \( Q = Z^{-1} \) where \[ Z = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \]

and with \( \Gamma_0 = \text{diag}(1, 2, 3), \Gamma_1 = I_3 \). With reference to equation (30), \[ C = \frac{1}{4} \begin{bmatrix} 15 & 3 & -1 \\ 5 & -3 & -3 \\ 3 & -1 & 3 \end{bmatrix} \]

and since \( C^\top \) is non-singular we have \[ \text{diag}(x_0)y_0 = C^{-\top} g = -\frac{1}{16} \begin{pmatrix} 3 \\ 6 \\ 15 \end{pmatrix} =: f. \]

Since none of the components of the right-hand side are zero all components of \( x_0 \) and \( y_0 \) have to be non-zero. Hence for all \( x \) with \( e_i^\top Z^{-1} x \neq 0 \) for all \( i = 1, 2, 3 \), \( \text{diag}(x_0) \) is non-singular and according to equation (34) we may write \[ y_0 = -\frac{1}{16} \text{diag}(x_0)^{-1} \begin{pmatrix} 3 \\ 6 \\ 15 \end{pmatrix} = -\frac{1}{16} \begin{pmatrix} 3/x_{01} \\ 6/x_{02} \\ 15/x_{03} \end{pmatrix}. \]

Defining the vector \( x_0^{-1} := [1/x_{01}, 1/x_{02}, 1/x_{03}]^\top \) the back-transformation yields \[ y = Z^{-\top} y_0 = Z^{-\top} \text{diag}(f)x_0^{-1} = \frac{1}{64} \begin{bmatrix} -1 & 10 & -35 \\ 4 & -16 & 20 \\ -5 & 2 & 5 \end{bmatrix} x_0^{-1} \]

which gives all solutions \( y \) in terms of all \( x_0 = Z^{-1} x \) with non-zero components. Indeed the particular choice \( x = e_3 = Zx_0 \) yields \( x_0 = Z^{-1} e_3 = [5, -1, -1]^\top/12 \). Thus \( y_0 = 3[-3/5, 6, 15]/4 \) and the back-transformation yields \( y = Z^{-\top} y_0 = -[-93, 12, 30]^\top/20 \).  

We will now derive a relation between the result given by Theorem 1 and the original problem described by equation (22). In order to obtain a relation to the original unit-rank modification we rewrite equation (31). In the light of equation (26) we find \[ \text{diag}(x_0)y_0 = \text{diag}(Z^{-1} x)Q^{-\top} y \]

\[ = \sum_{i,k=1}^n \text{diag}(Z^{-1} e_i)Q^{-\top} e_k x_i y_k \]

\[ = P_e^\top (x \otimes y) \quad \text{(41)} \]

where \( \otimes \) is the Kronecker product and where the \( n^2 \times n \) matrix \( P_e \) is given by \[ P_e^\top := [\text{diag}(Z^{-1} e_1)Q^{-\top} e_1, \text{diag}(Z^{-1} e_1)Q^{-\top} e_2, \ldots, \text{diag}(Z^{-1} e_n)Q^{-\top} e_n]. \]

Equation (31) now reads (see equation (21)) as \[ g = F^\top (x \otimes y) \quad \text{(43)} \]
\[ F := P_C A^T \in \mathbb{C}^{n^2 \times m}. \]  
(44)

To ensure a real-valued solution we expand (43) by doubling the order
\[ g_R = \bar{F}_R (x \otimes y). \]  
(45)

This equation is identical to equation (22) and represents a linear problem with respect to \( x \otimes y \). The difference to formulation (31) lies in the dimension: whilst the solution of (45) is \( n^2 \)-dimensional the solution of (31) is only \( n \)-dimensional. Of course both solutions are subject to constraints, and these are discussed in detail in a later section.

Although Theorem 1 provides an analytic discussion of the solutions of the pole placement for a wide range of linear dynamic systems, Theorem 1 is not applicable to the placement of zeros in general because the corresponding minors may be defective and hence not diagonalisable. This issue is addressed in the following section.

### 5. A PLACEMENT THEOREM FOR DEFECTIVE MINORS

For defective minors of simple systems Theorem 1 is not applicable and we have to find an alternative to simplify the basic equation (14). Suppose \( A(s) \) is simple but the minor \( A(s)^{\ell j k} = S_i^T A(s) S_k \) is defective and we want to place the zeros \( \lambda_j, j \in \bar{m} \) of this minor. Then equation (14) reads as
\[
g_{ik} = T (x^{(i)}) (y^{(k)}) \]  
(46)

where the \( j \)th element of \( g_{ik} \in \mathbb{C}^m \) is \( \text{det}(A(\lambda_j)^{\ell j k}) \) and where the superscript \( (i) \) on \( x \) denotes the deletion of the \( i \)th component, and similarly for \( y \). In this case the definition of \( T \) is the same as in (15) except that the matrices on the right-hand side of (13) are now given by
\[
H_{i j k} := \lambda_j (A(\lambda_j)^{\ell j k})^{\text{ad}}. \]  
(47)

We recall that the adjugate of an \( (n - 1) \times (n - 1) \) matrix is related to its \( (n - 2) \)th compound matrix and that the compound matrix of a diagonal matrix is also diagonal [22–24]. We also recall the Binet–Cauchy Theorem [22] which states that the compound matrix of a product is equal to the product of the compound matrices. In the light of equation (23) we may rewrite the adjugate in equation (47) as
\[
(A(\lambda_j)^{\ell j k})^{\text{ad}} = (S_i^T A(\lambda_j) S_k)^{\text{ad}} \]  
(48)

with the matrices \( V_i := \epsilon_{n-2} (Z^T S_i) J \), \( W_i := \epsilon_{n-2} (Q S_k) J \in \mathbb{C}^{n_2 \times (n-1)} \) and where the vector \( \ell_j \in \mathbb{C}^{n_2} \) is the diagonal of the diagonal matrix \( \epsilon_{n-2}(Y_j) \). The matrix \( J \) is defined [22–24] as \( J := E \Sigma \) where \( E \) is the rotated identity matrix,
\[
E := \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ 1 & 0 \end{bmatrix} \]  
(49)

and \( \Sigma = \text{diag}((-1)^j)_{j \in \bar{n}} \).
Suppressing the indices $i,k$ we define the vectors
\[ x_V := V_i x^{(i)}, \quad y_W := W_k y^{(k)} \in \mathbb{C}^{n_2} \] (50)
and the matrix (see equations (31) and (30))
\[ C_d := [\ell_1, \ldots, \ell_m] \in \mathbb{C}^{m \times m}. \] (51)
Equation (46) then becomes
\[ g_{ik} = A^t C_d \text{diag}(x_V) y_W. \] (52)
We have derived a result which is formally equivalent to equation (31) and may be summarised in the following theorem.

**Theorem 2.** Let the system described by $A(s) = \sum_{j=0}^d s^j A_j$ be simple and let the minor $(i,k)$ of $A(s)$ be defective. Then for fixed $\ell \in \mathcal{N}_0$ the disjoint complex numbers $\lambda_i, \ell \in \mathcal{N}_0$ are zeros of the minor $A(s)^{(ik)} + s^j x^{(i)} y^{(k)}$ if equation (52) has a solution $x_V, y_W$ such that the two linear problems
\[ x_V = V_i x^{(i)} \] (53)
\[ y_W = W_k y^{(k)} \] (54)
have solutions $x^{(i)}, y^{(k)} \in \mathbb{R}^{n_1}$.

Note that the solution of the non-linear problem (14) is now separated into a sequence of linear problems. Although equations (31) and (52) are formally equivalent the difference between these problems is the back-transformation. In the first case the transformation is unique whilst in the second case relations (53) and (54) represent two overdetermined systems of equations.

Theorem 2 can also be used to place zeros of a non-proportionally damped system. The difference here is the choice of the selecting matrices $S_i, S_k$ for a simple matrix pencil described by equations (3) or (4). The following example clarifies the situation.

**Example 3.** Let $M = I_3, K := \text{diag}(1, 4, 1)$ and
\[ D := \frac{1}{3} \begin{bmatrix} 4 & -\sqrt{5} & -1 \\ -\sqrt{5} & 8 & 0 \\ -1 & 0 & 1 \end{bmatrix} \]
then the system described by $A(s) := s^2M + sD + K$ is simple but its first principal minor $A_{(33)}(s)$ is defective with the repeated conjugate pair of eigenvalues $-1 \pm j$. We seek a symmetric unit-rank modification $sxy^\top$, $y = ax \in \mathbb{R}^3, 0 \neq a \in \mathbb{R}$, of the damping matrix such that the first principal minor of the modified system has the two conjugate pairs of zeros $(-0.294 \pm 0.9558 j, -0.9607 \pm 1.7542 j)$ which are the poles of the system $M_0 = I_2, K_0 := \text{diag}(1, 4)$ and $D_0 := \text{diag}(4 - \sqrt{5}, 8 - \sqrt{5})/3$. Since $D$ represents a non-proportional general viscous damping the three matrices $M, D, K$ are not diagonalisable simultaneously. However the pencil $A_S(s) := sA_1 + A_0$ with
\[ A_1 = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \in \mathbb{R}^{6 \times 6} \]
is diagonalisable because $A_S(s)$ is simple. Since the matrices $A_0, A_1$ are real-valued and symmetric there exist a non-singular matrix $X \in \mathbb{C}^{6 \times 6}$ such that
\[ X^\top A_1 X = I_6, \quad X^\top A_0 X =: \Omega = \text{diag}_{i \in \mathbb{N}}(\omega_i). \]
With reference to Theorem 1 we have $Z = X^{-1}$, $Q = X^{-1}$. The placement of the zeros $\lambda_1 = -0.294 + 0.9558i$, $\lambda_2 = -0.9607 + 1.7542i$ of the first principal minor of $A(s)$ by the modification $x_2x_3^{(3)}x_4^{(3)^T}$, $x^{(3)} \in \mathbb{R}^2$, is equivalent to placing the poles $\lambda_1, \lambda_2$ of $S_{3,6}^T A(s) S_{3,6}$ by the unit-rank modification

$$
\alpha \begin{pmatrix} x \\ 0 \end{pmatrix} (x^T, 0) \in \mathbb{R}^{6 \times 6}
$$

(55)
of $A_1$ where $S_{3,6} = [e_1, e_2, e_4, e_5] \in \mathbb{N}^{6 \times 4}$. Supposing the indices $(i, k)$ the matrices $V$ and $W$ of Theorem 2 are of size $20 \times 4$ and given by

$$
V \equiv W := \mathcal{C}_3 (X^{-1} S_{3,6}) J.
$$

(56)
Moreover we have $g^T = [1.4794 - 1.5787j, -4.2890 + 0.5730j] \in \mathbb{C}^2$. The matrix $C_d$ is $20 \times 2$ and has full rank. Hence the kernel of $C_d^T$ has dimension 18. If we denote by $N_C \in \mathbb{C}^{20 \times 18}$ a matrix whose columns span the kernel of $C_d^T$ then there are 18 arbitrary parameters $\eta \in \mathbb{C}^{18}$ for the solution

$$
\alpha \text{diag}(x_V) x_V = C_d^T A^{-1} g + N_C \eta.
$$

Since $\alpha \neq 0$ the $i$th component of this equation corresponds to (see equation (36))

$$
x_{V_i}^2 = \frac{1}{\alpha} (h_i + e_i^T N_C \eta) =: z_i^T p
$$

where $z \in \mathbb{C}^{19}$ is defined elementwise by $z_i := h_i, [z_2, \ldots, z_{18}] := e_i^T N_C$, and where $p^T := (1, \eta^T)/\alpha$. Equation (5) has the two solutions

$$
x_{V_i} = x_{V_i}(p) = \pm \sqrt{z_i^T p}
$$

for arbitrary $p$ with a real-valued non-zero first component. The problem here is the back-transformation. In the light of equations (55) and (56) we have

$$
x_V = V \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = [Ve_1, Ve_2] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =: V x^{(3)} \in \mathbb{C}^{20}
$$

and hence we have to solve the problem

$$
x_V(p) = V x^{(3)}
$$

for $x^{(3)} \in \mathbb{R}^2$. To ensure a real-valued solution we double the order of the above equation to give

$$
z(p) := \begin{pmatrix} \text{Re}\{x_V\}(p) \\ \text{Im}\{x_V\}(p) \end{pmatrix} = \begin{pmatrix} \text{Re}\{V x\} \\ \text{Im}\{V x\} \end{pmatrix} x^{(3)}
$$

$$
=: U
$$

which has the solution

$$
x^{(3)} = x^{(3)}(p) = U^+ z(p)
$$

(57)
if and only if $p \in \mathbb{R}^+ \times \mathbb{C}^{18}$ can be chosen in such a way that $z(p) \in \text{span}(U)$ or equivalently

$$
N_U z(p) = 0
$$

(58)
where $N_U := I_{40} - UU^+$. Equation (58) is formally equivalent to problem (40). The numerical evaluation of equation (58) reveals that there is only one unique solution
\[ p = e_1/x, \text{ with } x \approx -\sqrt{5/3} \text{ which leads via equation (57) to the unit-rank modification} \]

\[ x^{(3)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

This solution corresponds to the modified damping matrix

\[ D = \frac{\sqrt{5}}{3} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \beta e_3 e_3^\top = \frac{1}{3} \text{diag}(4 - \sqrt{5}, 8 - \sqrt{5}) + \beta e_3 e_3^\top \]

for arbitrary \( \beta \in \mathbb{R} \).

Again we would like to obtain an explicit relation analogous to equation (21). We note that the deletion of the \( i \)th element of the vector \( x \) can be written as \( x^{(i)} = S_i^\top x \). Hence with equation (50) we may write

\[ \text{diag}(x) y_W = \text{diag}(V_i S_i^\top x) W_k S_k^\top y = P_{ik} (x \otimes y) \]

where \( P_{ik} \in \mathbb{C}^{n \times n_2} \) is defined analogously to equation (42) by

\[ P_{ik} = [\text{diag}(V_i S_i^\top e_1) W_k S_k^\top e_1, \text{diag}(V_i S_i^\top e_1) W_k S_k^\top e_2, \ldots, \text{diag}(V_i S_i^\top e_n) W_k S_k^\top e_n]. \]

Finally, we rewrite equation (52) equivalently as

\[ g_{ik} = F_{ik}^\top (x \otimes y) \]

where

\[ F_{ik} := P_{ik} C_d A^t \in \mathbb{C}^{n^2 \times m}. \]

To ensure a real-valued solution we rewrite equation (61) as

\[ g_{Rik} = F_{Rik}^\top (x \otimes y) \]

where, analogous to equation (45),

\[ g_{Rik} := \begin{pmatrix} \text{Re}\{g_{ik}\} \\ \text{Im}\{g_{ik}\} \end{pmatrix} \in \mathbb{R}^{2m} \]

\[ F_{Rik} := [\text{Re}\{F_{ik}\}, \text{Im}\{F_{ik}\}] \in \mathbb{R}^{n^2 \times 2m}. \]

In the light of equations (63) and (45), the formal equivalence of the placement of poles or zeros is now obvious. Although these equations show an explicit relation to the original modification there may be additional constraints in the solution. How to solve non-linear problems of this type by incorporating additional information is explored in the following section.

6. CONSTRAINT SOLUTIONS

In the previous sections we have derived equations for the placement of poles and zeros of simple matrix pencils. We have discussed several methods of solving these equations and we have shown that the problems of placing poles or zeros are formally equivalent (see also equation (21)). In this section, we seek to simplify the equations by incorporation of additional information. We focus our attention on the problem of solving

\[ g = F^\top \phi \]
for a given vector \( g \in \mathbb{R}^{n_f} \) and a given matrix \( F \in \mathbb{R}^{n \times n_f} \) subject to
\[
\phi = x \otimes y \in \mathbb{R}^n.
\]  
(67)

Equation (66) is linear with respect to \( \phi \in \mathbb{R}^n \) and has a solution if
\[
g \in \text{span}(F^\top).
\]  
(68)

However this condition is necessary but not sufficient. To emphasise this, suppose equation (68) holds. Moreover let the columns of \( G \in \mathbb{R}^{2m \times n_f} \) form a basis of the kernel of \( F^\top \) then
\[
\phi(\eta) := \underbrace{F^\top g + G\eta}_{=: h} \in \mathbb{R}^n
\]  
(69)
solves equation (66) for all \( \eta \in \mathbb{R}^{nf} \). Let \( \phi(\eta) \) be the sequence of columns of \( \Phi(\eta) \in \mathbb{R}^{n \times n} \) then equation (67) is equivalent to
\[
xy^\top = \Phi(\eta) = H + \sum_{i=1}^{n_f} G_i \eta_i
\]  
(70)

where \( h \in \mathbb{R}^{n_f} \) contains the sequence of columns of \( H \in \mathbb{R}^{nf \times n} \) and where for every \( i \in \tilde{\mathfrak{n}}_f \) the sequence of columns of \( G_i \in \mathbb{R}^{n \times n_f} \) form the \( i \)th column of \( G \). Since the left-hand side of (70) has unit-rank a solution exists only if \( \eta_i \) can be found such that the right-hand side also has unit rank. We summarise our results by the following

**Lemma 1.** If the necessary condition equation (68) holds true then a real-valued unit-rank modification \( xy^\top \) satisfying equation (66) exists if and only if \( \eta \in \mathbb{R}^{nf} \) can be found such that \( \Phi(\eta) \) defined by equation (70) has unit rank.

To obtain all solutions of equation (70) is not a trivial task if the dimension \( n_f \) is large. However, in practical applications there are usually constraints on \( x, y \) which can be used to reduce the dimensionality of the solution. We will now incorporate information which is relevant for a wide range of dynamic systems.

6.1. **Symmetric Unit Rank Modifications**

We require that the unit-rank modification \( xy^\top \) is symmetric. In this case \( y = \alpha x \) for some non-zero \( \alpha \in \mathbb{R} \). Then equation (70) becomes
\[
xx^\top = \frac{1}{\alpha} H + \sum_{i=1}^{n_f} G_i \frac{\eta_i}{\alpha} =: \sum_{i=1}^{n_H} H_i \xi_i
\]  
(71)

where \( n_H := n_f + 1 \) and where
\[
(\xi_1, \xi_2, \ldots, \xi_{n_H}) := \frac{1}{\alpha} (1, \eta_1, \ldots, \eta_{n_f})
\]  
(72)

\[
(H_1, H_2, \ldots, H_{n_H}) := (H, G_1, \ldots, G_{n_f}).
\]  
(73)

In general the matrices \( H_i, i \in \tilde{\mathfrak{n}}_h \), are not symmetric. Hence the symmetry conditon of the right-hand side of equation (71) requires
\[
\sum_{i=1}^{n_H} (H_i - H_i^\top)\xi_i = 0
\]  
(74)
or equivalently
\[ S \zeta = 0 \]  \hspace{1cm} (75)
where the \( i \)th column of \( S \in \mathbb{R}^{n^2 \times n_H} \) consists of the series of the columns of \( H_i - H_i^T \). If we denote by \( N \in \mathbb{R}^{n_H \times n_S} \) a matrix which columns form a basis of the kernel of \( S \) then the solution of equation (75) is
\[ \zeta = N \theta. \]  \hspace{1cm} (76)
Inserting this result into equation (71) yields
\[ x x^T = \sum_{i=1}^{n_H} H_i e_i^T N \theta \]
\[ = \sum_{k=1}^{n_S} \sum_{i=1}^{n_H} H_i e_i^T N e_k \theta_k \]
\[ = \sum_{k=1}^{n_S} R_k \theta_k. \]  \hspace{1cm} (77)
Note that in the light of equation (72) a solution \( \theta \) of equation (77) is a solution of equation (71) if and only if
\[ 1/x = \xi_1 = e_1^T N \theta \neq 0. \]  \hspace{1cm} (78)
This condition may lead to a further reduction of the dimension of the solution space.

6.2. STRUCTURE CONSISTENT SOLUTIONS

In practical applications the vector \( x \) is always related to certain model substructures which represent realisable system modifications, for instance additional lumped masses, single springs or dashpots. These unit-rank substructures can be represented by a finite set of given vectors
\[ \mathbb{S} := \{b_1, \ldots, b_s\}, \quad b_i \in \mathbb{R}^n. \]  \hspace{1cm} (79)
Any permissible modification \( x x^T \) has to be generated by a vector of \( \mathbb{S} \), so that for a fixed \( i \in \mathbb{S} \)
\[ x = b_i. \]  \hspace{1cm} (80)
For example consider the vibrational chain described by \( M = \text{diag}(m_1, m_2, m_3) \) and
\[ K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix}. \]  \hspace{1cm} (81)
Then for every stiffness modification \( x x^T \) the vector \( x \in \mathbb{R}^3 \) has to have one of the following forms:
\[ \mathbb{S} := \{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}. \]  \hspace{1cm} (82)
The case \( x = e_i, i \in \mathbb{S} \), corresponds to a grounded spring connected to the \( i \)th mass, whilst the case \( x = e_i - e_k, i \neq k \), corresponds to a spring between the \( i \)th and the \( k \)th masses. If \( x \notin \mathbb{S} \) then the corresponding modification cannot be realised with a single spring modification, although it is possible to combine a number of springs to produce a unit-rank modification.
To incorporate the knowledge of the underlying model structure we insert equation (80) into equation (77) and find

$$b_i b_i^\top = \sum_{k=1}^{n_S} R_k \theta_k$$

or equivalently

$$p_i = R \theta$$

where $p_i := b_i \otimes b_i$ and where the $k$th column of $R \in \mathbb{R}^{r^2 \times n_S}$ consists of the sequence of columns of $R_k$. Equation (84) is equivalent to

$$[p_i, R] \begin{pmatrix} -1 \\ \theta \end{pmatrix} = 0$$

and has a solution if and only if the kernel of $[p_i, R]$ is generated by at least one vector with a non-zero first component, which in addition satisfies condition (78).

To clarify the results we study the simultaneous pole–zero placement by a unit-rank stiffness modification $\alpha x x^\top$, $0 \neq \alpha \in \mathbb{R}$, of the 3 dof vibrational chain example introduced above.

**Example 4.** Consider the vibrational chain defined above with equal masses $m_1 = m_2 = m_3 = 1$ kg and equal spring stiffnesses $k_1 = k_2 = k_3 = 1$ N/m. The absolute value of the six different elements of the transfer matrix is plotted in Fig. 2. For motivation we assume the following scenario. At mass $m_2$ the system is subjected to a random excitation that produces a response spectrum with a maximum at about $f = \sqrt{2}$ rad/s, corresponding to $\lambda = 2$. The mass $m_2$ represents equipment that is sensitive to excitation frequencies in a neighbourhood of $\omega$. The equipment located at the remaining dofs is insensitive. The goal is to find a symmetric unit-rank modification of the stiffness matrix which (see Fig. 2)

1. preserves zero 2 of minors $(1, 2)$ and $(2, 2)$,
2. places zero 2 of minor $(2, 3)$ and
3. places pole 2 of the system.

The idea is that the simultaneous preservation and the placement of the same pole and zero will lead to a pole–zero cancellation of all minors involving dof 2 whilst the remaining dof will have a resonance at the cancelled pole. Thus the main energy flow is shifted to the insensitive parts of the structure.

The above conditions correspond to four equations of type

$$g_i = f_i^\top (x \otimes x) x$$

with $g_i \in \mathbb{R}$ and $f_i \in \mathbb{R}^g$ for $i \in \mathcal{I}$, respectively. Collecting all contributions we find

$$g = (0, 0, -1, -1)^\top$$

and

$$F^\top = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ -2 & 0 & 2 & 0 & 0 & -2 & 2 & -2 & -2 \end{bmatrix}$$
Figure 2. Example 4: Logarithm of the absolute values of the 6 different elements of \((K - \lambda M)^{-1}\) for \(\lambda \in [0, 4]\).

The matrix \(F\) has full rank. Hence the kernel of \(F^\top\) is five-dimensional. If the kernel is spanned by the columns of \(G \in \mathbb{R}^{9 \times 5}\) then we have

\[
x(\otimes x) = h + G\eta
\]

where a brief calculation reveals

\[
h = F^+ g = e_1 \in \mathbb{R}^9.
\]

According to equation (71) we may write equivalently that

\[
x x^\top = \frac{1}{x} e_1 e_1^\top + \sum_{i=1}^{5} G_i \frac{\eta_i}{x}.
\] (86)
None of the five matrices $G_i$ is symmetric. Hence we reduce the dimension of the solution space by the condition

$$\sum_{i=1}^{5} (G_i - G_i^\top)\eta_i = 0$$

which is equivalent (see equation (75)) to

$$S\eta = 0.$$ 

The matrix $S \in \mathbb{R}^{9 \times 5}$ has rank 3 and hence there exists a matrix $R \in \mathbb{R}^{5 \times 2}$ whose columns form a basis of the kernel of $S$. If we insert

$$\eta = R\theta$$

Figure 3. Example 4: Logarithm of the absolute values of the 6 different elements of $(K + zzz^\top - \lambda M)^{-1}$ for $\lambda \in [0, 5]$. 
into equation (86) we find, according to equation (77), that

\[ xx^\top = \frac{1}{\lambda} e_1 e_1^\top + R_1 \frac{\theta_1}{\lambda} + R_2 \frac{\theta_2}{\lambda} \]

with

\[
R_1 = \begin{bmatrix}
0 & 0.5561 & 0 \\
0.5561 & -0.6177 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
R_2 = \begin{bmatrix}
0 & -0.4368 & 0 \\
-0.4368 & -0.7864 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We proceed with the incorporation of the structural information. Every permissible stiffness modification has to be generated by vectors \( x \in \mathbb{S} \), defined in equation (82). From the computation of equation (85) for each of the six vectors of \( \mathbb{S} \) we find that solutions exist only for the three cases \( e_1, e_2, e_1 - e_2 \) which are given by,

\[
\frac{1}{\lambda} \begin{pmatrix}
1 \\
\theta_1 \\
\theta_2
\end{pmatrix} \in \left\{ \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
-0.6177 \\
-0.7864
\end{pmatrix}, \begin{pmatrix}
1 \\
-1.7299 \\
0.0871
\end{pmatrix} \right\}
\]

respectively. We have to exclude the second solution (see equation (78)) because it would correspond to \( 0 = 1/\lambda \). The two remaining solutions correspond to \( \lambda = 1 \) with

\[ x \in \{ e_1, e_1 - e_2 \} \]

The first solution corresponds to an additional grounding spring. If we exclude this case the remaining solution leads to the modified stiffness matrix

\[ K_m := K + (e_1 - e_2)(e_1 - e_2)^\top \]

which corresponds to an increase of the stiffness of the spring between \( m_1 \) and \( m_2 \). The absolute value of the 6 different elements of the transfer matrix of the modified system are shown in Fig. 3. Clearly an excitation at \( m_2 \) with maximum spectrum about \( \omega = \sqrt{2} \text{rad/s} \) will cause a response at \( m_2 \) with negligible magnitude.

7. CONCLUSIONS

For general simple matrix pencils we have derived the fundamental equations for the problems of placing poles or zeros by unit-rank modifications. Both problems are formally equivalent to an equation which is difficult to solve numerically for all solutions, in general. Several methods have been suggested in order to obtain an overview of the entire solution space. We have clarified our results by several numerical examples.

The examples given have been based on discrete models with a low number of dofs. Furthermore, the modifications have either been quite general, with no consideration of how the structure might be physically modified, or require the application of constraints so that the modifications should relate to a single spring. The authors are continuing to investigate applications of the proposed pole–zero placement approach, and one of these applications is considered in more detail here.
Often the measured data is difficult to correlate to the corresponding finite element model, particularly in the case of cyclically symmetric and axisymmetric structures. Close or repeated modes are very sensitive to small changes in the structure, making correlation of these modes difficult. Mottershead et al. [25] suggested that fictitious modifications may be applied to both the measured data set and the model in order to separate these close modes and allow better correlation. Their modifications were single discrete springs or masses, but the methods proposed in this paper open up a wider range of possible modifications. Since the modifications do not have to be physically applied, the realisability of the unit-rank modification is not an issue.

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REFERENCES


APPENDIX A: NOMENCLATURE

\[\mathbb{N}\] set of natural numbers
\[\mathbb{R}\] set of real numbers
\[\mathbb{C}\] set of complex numbers
\[\mathbb{n}\] := \{1, \ldots, n\} for all \(n \in \mathbb{N}\)
\[\mathbb{n}_0\] := \{0, 1, \ldots, n\} for all \(n \in \mathbb{N}\)
\[\text{diag}(c)\] := \text{diag}_{\mathbb{n}}(c_i) a diagonal matrix with the vector \(c\) on its diagonal
\[I_n\] identity matrix of order \(n \in \mathbb{N}\)
\[e_i\] vector of appropriate dimension with zeros everywhere except a 1 in component \(i\)
\[A^{(\mathcal{I}|\mathcal{K})}\] matrix that results from \(A \in \mathbb{C}^{n \times m}\) by deleting all rows with indices in \(\mathcal{I} \subseteq \mathbb{n}\) and all columns with indices in \(\mathcal{K} \subseteq \mathbb{m}\)
\[S_{\mathcal{I}}\] := \(I_{\mathbb{n}}^{(\mathcal{I} \times \mathcal{I})}\) selecting matrix resulting by deletion of all columns of \(I_{\mathbb{n}}\) with indices in \(\mathcal{I} \subseteq \mathbb{n}\)
\[A^{\text{adj}}\] adjugate of the matrix \(A\)
\[x^{(i)}\] := \(S_{\mathbb{n}}^{(i)} x\) vector that results from the vector \(x\) by deleting the \(i\)th component
\[(A)_{i,k}\] element in row \(i\) and column \(k\) of \(A\)
\[n_m\] := \(n!/(m!(n-m)!}\) number of combinations without repetitions of \(m\) out of \(n\)
\[C_m(A)\] \(m\)th compound matrix of \(A\)
\[\otimes\] Kronecker product