Robust balancing for rotating machines

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Abstract: The balancing of rotors divides broadly into two categories: balancing in situ and balancing in a balancing machine. In the latter case, the most common practice is to arrange balance corrections on the rotor such that the net excitations of each of the four in-plane rigid-body modes of the free rotor is zero by deploying balance corrections on two independent planes. In a small proportion of cases, the net excitations of the first pair of flexural modes are also zeroed using a third correction plane.

This paper proposes that, when rotors are balanced in a balancing machine (not similar to the machine stator), substantially more utility can be gained from the balancing operation by combining a suitably weighted account of the specific balancing requirements of the machine with knowledge of the expected machine characteristics than can be achieved by ignoring this knowledge. A single cost function is established based on a numerical model of the machine. Then, depending on circumstances, either the expected value of this cost function or its worst possible value can be minimized.

The methods proposed require that relatively detailed knowledge of the distribution of residual unbalance be obtained experimentally. The paper briefly discusses some practical methods for how such information might be extracted. The definition of the cost function as a matrix quadratic form provides potentially valuable information about the necessary number and the optimal location of balance planes on a given rotor, and methods for determining an optimal set of balance planes are outlined.

Keywords: balancing, rotor dynamics, rotating machines, robustness, distributions of unbalance

NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>(a_k)</td>
<td>weighting factor associated with the (k)th vibration output</td>
</tr>
<tr>
<td>(A)</td>
<td>((P \times P)) diagonal matrix of weighting factors relating cost to (y) [see equation (4)]</td>
</tr>
<tr>
<td>(A')</td>
<td>augmented version of (A) for numerous different speeds [see equation (9)]</td>
</tr>
<tr>
<td>(b)</td>
<td>complex (N)-dimensional vector representing forces required to reproduce static rotor bend</td>
</tr>
<tr>
<td>(b_R, b_S)</td>
<td>partitions of (b) corresponding to the rotor and stator degrees of freedom</td>
</tr>
<tr>
<td>(c)</td>
<td>complex (N)-dimensional vector corresponding to correction unbalance</td>
</tr>
<tr>
<td>(c_R, c_S)</td>
<td>partitions of (c) corresponding to the rotor and stator degrees of freedom</td>
</tr>
<tr>
<td>(C)</td>
<td>((N \times N)) system damping matrix for the rotating machine</td>
</tr>
<tr>
<td>(d)</td>
<td>complex (M)-dimensional vector specifying correction unbalance at the correction planes</td>
</tr>
<tr>
<td>(D)</td>
<td>complex ((P \times M)) matrix relating the (M)-dimensional vector of unbalance correction quantities to changes in the (P)-dimensional vector ((Ay)) according to equation (7)</td>
</tr>
<tr>
<td>(D')</td>
<td>augmented version of (D) for numerous different speeds</td>
</tr>
<tr>
<td>(DOF)</td>
<td>degrees of freedom</td>
</tr>
<tr>
<td>(E)</td>
<td>complex positive-definite Hermitian ((M \times M)) matrix</td>
</tr>
<tr>
<td>(E)</td>
<td>Young’s modulus</td>
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F distributions of residual unbalance according to equation (18)
g $M$-dimensional vector of \( E \)
\( \mathbf{G} \) \((N \times N)\) system gyroscopic matrix for the rotating machine
\( h \) scalar
\( j \) imaginary number \( \sqrt{-1} \)
\( J \) cost associated with a particular state of unbalance and bend
\( J' \) augmented cost for many speeds or parameter values
\( \mathbf{K} \) \((N \times N)\) system stiffness matrix for the rotating machine
\( L \) number of effective balance planes
\( M \) number of balance planes available on the rotor
\( \mathbf{M} \) \((N \times N)\) system mass matrix for the rotating machine
\( N \) number of degrees of freedom in the complete machine model
\( p_i \) weighting of the uncertain parameter values
\( P \) number of distinct vibration measures used to determine overall cost
\( q \) full-length complex vector of synchronous vibration deflections
\( q_R, q_S \) partitions of \( q \) corresponding to the rotor and stator degrees of freedom respectively
\( Q \) number of discrete speeds taken into account in the robust balancing
\( R \) number of uncertain parameters in the machine model
\( \mathbf{R} \) \((P \times N)\) selection matrix which relates \( \mathbf{y} \) to \( q \) [see equation (5)]
\( \mathbf{S} \) \((N \times M)\) selection matrix which relates \( e \) to \( d \) [see equation (3)]
\( \mathbf{u} \) complex \( N \)-dimensional vector corresponding to the original state of unbalance
\( u_R, u_S \) partitions of \( u \) corresponding to the rotor and stator degrees of freedom
\( \mathbf{U} \) left modal matrix for the undamped system \( \mathbf{K}, \mathbf{M} \)
\( \mathbf{V} \) right modal matrix for the undamped system \( \mathbf{K}, \mathbf{M} \)
\( \mathbf{y} \) complex \( P \)-dimensional vector containing the \( P \) measures of vibration contributing to the cost
\( y' \) augmented version of \( y \) for numerous different speeds [see equation (9)]
\( z \) complex \( P \)-dimensional vector value held by \( (\mathbf{A}y) \) when no correction has been applied
\( z' \) augmented version of \( z \) for numerous different speeds
\( \mathbf{T} \) effectiveness of the distributions of residual unbalance according to equation (18)
\( \theta_i \) \( i \)th uncertain parameter
\( \Theta_i \) set of values the \( i \)th uncertain parameter can take
\( \mathbf{\Lambda} \) diagonal matrix of natural frequencies squared
\( \Phi \) modal damping matrix
\( \Omega \) rotor spin speed \((\text{rad/s})\)

1 INTRODUCTION

Unbalance forces arise on the rotors of all rotating machines as a result of imperfect symmetry in the distribution of rotor mass. In a perfectly balanced rotor, the centre of gravity of every thin slice of the rotor normal to the axis of rotation would lie on that axis. Even very small distances between the actual centre of gravity of one thin slice of a rotor and the axis of rotation can cause relatively large forces for high rotational speeds. Balancing is the process of deliberately adjusting the distribution of unbalance on a rotor such that the forces acting as a result of the residual unbalance (and perhaps from other sources also) have less negative effects.

The literature on balancing is large. Parkinson [1] provides a reasonably comprehensive review including more than 60 references on the subject prior to 1990. Foiles et al. [2] provide a more recent review containing more than 160 references prior to 1998. There are many different concerns with balancing. Balancing with a limited number of trials (or sometimes none) is important in many cases [3–6]. Balancing rotors that are not isotropic involve different considerations now being regularly addressed [7]. Computing suitable unbalance corrections subject to bounding constraints on the unbalance corrections is now perceived to be an issue [8], as is computing suitable unbalance corrections given multiple criteria [9].

Although it does not conform to the above definition for balancing, so-called ‘adaptive balancing’ where synchronous forces are applied between the rotor and stator by magnetic bearings or other active bearings (or bearing seatings) is increasingly subject to interest [10]. Artificial neural networks are beginning to be applied in rotor balancing [11, 12], further extending the range of activities falling under the general description of rotor balancing.

Balancing rotors, which are assembled from exchangeable modules, are addressed by Schneider in reference [13]. Balancing a flexible rotor at speeds well below the first critical speed is recognized as a challenge in cases where the rotor is mounted in its final stator...
Ehrich [15] addressed the issue of achieving a high effectiveness of balance combining knowledge of the rotor characteristics and the likely distributions of unbalance within the rotor.

This paper is concerned primarily with the balancing of rotors in balancing machines that are not dynamically similar to the stators in which these rotors will ultimately be deployed. Its contribution has some overlap with references [13, 14] and [15]. It presents methods whereby the following elements of knowledge are all combined to determine an optimal unbalance correction given a fixed set of balance planes:

(a) the expected distribution of rotor properties,
(b) the expected distribution of stator (and bearing) properties and
(c) a cost function reflecting the correct relative importance of the various balancing objectives.

Although numerous papers have incorporated some knowledge of machine dynamics or balancing cost, the present paper is distinguished from other papers in the balancing literature by the simultaneous utilization of all three of the above areas of knowledge. The method presented assumes initially that practical methods can be devised for determining the distributions of residual unbalance on a rotor to sufficiently high resolution. The highest speed at which a rotor can be spun in a balancing machine appears initially to place limits on that resolution. The final section of the paper provides some justification for the expectation that this determination is practicable.

2 MACHINE MODELS AND BALANCE OBJECTIVES

Interest in this paper is focused exclusively on harmonic responses at synchronous frequency (shaft speed), $\Omega$. Consider that a displacement coordinate system has been established for the rotating machine model such that the complex $N$-dimensional vector $q$ describes the complete set of synchronous deflections of the machine. This vector is a function of $\Omega$ since

$$(K - \Omega^2M) + j\Omega(C + \Omega G)q = \Omega^2(u + c) + b \tag{1}$$

Vector $u$ represents the original distribution of unbalance on the rotor, $c$ represents the corrective unbalance distribution applied to the rotor and $b$ represents the set of static forces necessary to reproduce any rotor bend present at vanishingly low shaft speed. Vectors $q$, $u$, $c$ and $b$ can each be considered to be partitioned into a rotor partition (subscript R) and a stator partition (subscript S) as

$$q = \begin{bmatrix} q_R \\ q_S \end{bmatrix}, \quad u = \begin{bmatrix} u_R \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} c_R \\ 0 \end{bmatrix}$$

$$b = \begin{bmatrix} b_R \\ 0 \end{bmatrix} \tag{2}$$

The stator partitions of $u$, $c$ and $b$ are all necessarily zero but $q_S$ is not generally zero since stator deflections will arise naturally as a result of forces exerted on the stator from the rotor.

Generally, the number of planes where balance correction may be applied is limited and this limitation may be expressed by writing the full vector of unbalance corrections, $c$, in terms of the vector $d$ containing the unbalance corrections at the available correction planes:

$$c = Sd \tag{3}$$

If a given rotor has $M$ balance planes, $d$ will have $M$ entries. The $(N \times M)$ selection matrix $S$ is real-valued and indicates how the forces at the individual balance planes are distributed on the rotor. This matrix can also be partitioned for consistency with equation (2), but this serves no purpose.

Figure 1 shows a simple rotor together with a global Cartesian axis set and also shows the direction of positive rotation. At the instant $t = 0$, some marker on the rotor is directly aligned with the global $x$ axis. The real part of $d$ contains, in order, the $x$ components of unbalance corrections at each of the balance planes when the rotor is in this reference position while the imaginary part of $d$ contains, in order, the negatives of the $y$ axis components of unbalance corrections at each of the balance planes at the same instant.

It is assumed that any vibration of the machine is undesirable but that there is, in general, some finite number, $P(< N)$, of different vibration measures contained in the complex vector $y$ that are significantly undesirable. Associated with these, there is a set of $P$ real constants, $a_k$, for $1 \leq k \leq P$ reflecting the relative undesirability of each one. For example, synchronous vibrations of a rotor at a plane where rotor–stator clearance is very small might be very undesirable.
whereas some bearing pedestal vibration on the same machine might be tolerable at much higher levels. The diagonal \((P \times P)\) matrix \(A\) having the values \(a_k\) in order on its diagonal then defines a quadratic cost function, \(J\). An illustration of this is given in Section 8. Through correct choice of the various \(a_k\), the minimization of \(J\) reflects the overall balancing objective:

\[
J = y^H A^2 y
\]  

where \(y^H\) is the Hermitian transpose of vector \(y\). This ensures that the cost function is always real-valued and positive. A linear relationship exists between \(y\) and \(q\) as

\[
y = Rq
\]

where \(R\) is a real-valued \((P \times N)\) matrix and the cost function can thus be rewritten as

\[
J = q^H (R^T A^2 R)q
\]

Note that since \(R\) is real-valued, \(R^T = R^H\).

For a given machine model, a given shaft speed, \(\Omega\), and a given initial state of rotor unbalance and bend, there is some combination of correction unbalances that minimizes \(J\). This combination of correction unbalances is the optimum for that particular machine at that state. It may not be unique.

### 3 INFLUENCE COEFFICIENTS AND MODAL BALANCING

The practice of balancing rotors using influence coefficients can be expressed clearly using the notation established above. Effectively, this method determines the relationship between correction unbalance terms and the final (weighted) synchronous outputs as:

\[
A y = [A R^T] (K - \Omega^2 M) + j \Omega (C + \Omega G) \begin{bmatrix} y(\Omega_1) \\ y(\Omega_2) \\ \vdots \\ y(\Omega_Q) \end{bmatrix} = z + Dd
\]

where the complex \(P\)-dimensional vector \(z\) is normally obtained directly from measurements \((y)\) on the uncorrected rotor \((d = 0)\) and the complex \((P \times M)\) matrix \(D\) may be obtained (because linearity is assumed) one column at a time from \(M\) other measurement runs in which the rotor carries trial masses or from a reliable numerical model. To minimize the cost function, \(J\), the following correction unbalance vectors are computed:

\[
d_{\text{optimal}} = (D^H D)^{-1} D^H z
\]

where the inverse is presumed to exist.

In equation (7), it is implied that only one spin speed is considered. However, it is well known [16] that there is no difficulty in combining data from \(Q\) different spin speeds by defining an augmented vector \(y'\) and an augmented weighting matrix \(A'\) as follows:

\[
y' = \begin{bmatrix} y(\Omega_1) \\ y(\Omega_2) \\ \vdots \\ y(\Omega_Q) \end{bmatrix}
\]

\[
A' = \begin{bmatrix} A(\Omega_1) & 0 & \cdots & 0 \\ 0 & A(\Omega_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\Omega_Q) \end{bmatrix}
\]

The augmented influence matrix \(D'\) and vector \(z'\) follow logically and equation (8) may be used directly with \(\{D', z'\}\) in place of \(\{D, z\}\).

Modal balancing of rotors is equally well expressed in this format. The modes in this case may be natural modes of the free–free rotor, natural modes of the rotating machine or, indeed, natural modes of a balancing machine with the rotor supported in it. In all cases, these natural modes are defined for some reference speed, \(\Omega_{\text{ref}}\), often taken as zero. The system matrices are assumed to be decoupled approximately by a real modal transformation, \(\{U, V\}\), as

\[
U^T K V = \Lambda \quad \text{(diagonal)}
\]

\[
U^T (C + \Omega_{\text{ref}} G) V \approx \Phi \quad \text{(diagonal)}
\]

\[
U^T M V = I
\]

where \(\Lambda\) and \(\Phi\) are real diagonal matrices. In general, this transformation is not achievable exactly but standard practice will be followed and the off-diagonal terms that occur in \(\Phi\) will be ignored.

Discarding the bend term \(b\), the equivalent of equation (7) for modal balancing becomes

\[
A y = \Omega^2 A R V \begin{bmatrix} \Lambda - \Omega^2 I \\ \Omega G \end{bmatrix}^{-1} [U^T (u + c)]
\]

Modal balancing seeks to zero the net excitation of a finite number of low-resonance frequency modes, by setting

\[
W^T (u + c) = 0
\]

where \(W\) comprises a submatrix of \(U\) containing the relevant (left) natural modes.

Substituting using equation (3) leads to the following expression for the correction unbalances for modal balancing:

\[
(W^T S) d = -W^T u
\]

The number of modal excitations to be cancelled is normally chosen such that \((W^T S)\) is square. In practice, modal balancing is normally approached one mode at a time. When the first four (rigid-body) modes of a free
rotor are chosen to be zeroed, this reproduces conventional two-plane balancing exactly.

The objective of this section has been to cast the most important existing methods in the framework set out in the previous section so that this framework can be seen most easily as a direct expression of established thinking in machine balancing.

4 A SIMULATED EXAMPLE

An example system is introduced here and the merits of influence coefficient balancing, modal balancing using the free–free modes of the machine and modal balancing using the natural modes of the machine itself are examined. The system is deliberately simple so that its outcomes can be reproduced readily. The same example is addressed again later in the context of methods that will be proposed.

Figure 2 shows the system being considered. It comprises a flexible rotor, supported on a flexible stator through flexible connections at three locations. The flexible stator is itself flexibly supported from rigid ground at its two ends. Both the rotor and stator are represented using simple Euler beam elements. The rotor carries two discs as shown in Fig. 2 and the stator carries four concentrated masses. The numerical data is as follows:

*Example system* (40 DOF, 26 on the rotor and 14 on the stator)

*Material:* $E = 209 \times 10^6 \text{ N/m}^2$, $\rho = 7850.0 \text{ kg/m}^3$

Length of each section of rotor shaft = 0.1 m

Rotor bar diameters = 0.08 and 0.14 m

Length of each section of stator beam = 0.2 m

Stator bar diameter = 0.25 m for all sections

Both rotor discs have o.d. of 0.5 m and thickness of 0.06 m

All rotor–stator springs have stiffness of $1.0 \times 10^8 \text{ N/m}$

Both stator–ground springs have stiffness of $1.0 \times 10^{10} \text{ N/m}$

Stator ‘end masses’ each have mass of 300 kg and inertia of 10.0 kg m$^2$

Stator ‘central masses’ each have mass of 200 kg and inertia of 0 kg m$^2$

Only the vertical vibrations of the system are considered. Any coupling between vertical and horizontal planes is ignored. The rotor has an initial state of unbalance caused by three discrete unbalances, as indicated in Fig. 2, and has four balance planes (BP1 to BP4), as indicated.

Figure 3 shows the first four (mass-normalized) natural modes of the free–free rotor. Figure 4 shows the steady state displacement response of one disc of the free–free rotor both before and after balance corrections have been added to the rotor on balance planes 1 and 2 such that the net excitation of the two rigid-body modes is zero. Evidently, this balancing makes a dramatic improvement at low frequencies, but at frequencies between 100 and 500 Hz the absolute displacement of the rotor at disc 1 is substantially worse after the balancing operation than before. The unbalance corrections applied to cancel out the excitation of the two rigid-body modes were 3.333333 and −5.333333 gm respectively (a positive unbalance being in the same direction as the 2 gm initial unbalance).

Figure 4 also shows the effect of using balance planes 1 to 3 to cancel out the net excitation of both the first
two rigid-body modes of the free–free rotor and the first flexural mode of the free–free rotor. Not surprisingly, there is no peak in response at the first flexural resonance frequencies for the balanced rotor. The unbalance corrections applied to cancel out the excitation of the two rigid-body modes and the first free–free mode were $-0.872622$, $-3.230356$ and $2.102978$ g m respectively.

In the present case, the modal properties of the free–free rotor are quite different from the modal properties of the assembled machine. Figure 5 shows the modal deflections (on the rotor only) as computed for the assembled machine.

Figure 6 shows the effect of having applied two- and three-plane modal balancing of the free rotor to the assembled machine. In this case, the quantity measured was selected as the relative motion between disc 1 of the rotor and the nearest point on the stator. The outcomes are surprising. In this case, applying conventional two-plane balancing on the free rotor makes this particular vibration measurement more than twice as bad at low speeds as it would be for the uncorrected rotor. Applying three-plane balancing on the rotor does reduce the vibration at low speeds but the reduction factor is less than two. Note that the vibration measurement being examined in this case is a relative motion between the

![Fig. 3 Mass-normalized mode shapes of the free–free rotor](image3)

![Fig. 4 The effect of two-plane and three-plane balancing on response of the free–free rotor](image4)
rotor (disc 1) and stator. If it had been an absolute stator vibration, then it would be expected that cancelling the net excitation of the free–free modes of the rotor would invariably have accounted for an improvement at low speeds.

Obviously, if the natural modes of the machine are used as the basis of modal balancing, the results are much better, as Fig. 7 shows. However, even here, there are some surprises. Balancing the rotor such that the net unbalance excitation of the first two machine modes was zeroed produced a very substantial benefit at low speeds (balance planes 1 and 2 were used). Extending this to zero, the net excitation of each of the first three modes (using balance planes 1 to 3) caused the particular vibration measurement of interest to worsen again at low speeds. The best low-speed results were achieved in this case by setting to zero the net excitation of machine modes 1, 2 and 4 using balance planes 1 to 3. Table 1 shows the unbalance corrections utilized in each case.

The clear message from the above instances of modal balancing is this:

Although it is clear that, if zero net unbalance excitation of all modes of either the free rotor or the assembled machine

![Mass-normalized mode shapes of the assembled machine (rotor deflections only)](image1)

**Fig. 5** Mass-normalized mode shapes of the assembled machine (rotor deflections only)

![Response (disc 1 relative to the stator) after balancing two and three modes of the free–free rotor](image2)

**Fig. 6** Response (disc 1 relative to the stator) after balancing two and three modes of the free–free rotor
could be obtained, the state of balance would be perfect, it is not by any means certain that zeroing the net unbalance excitation of any one particular mode will generally be advantageous.

Looking at influence coefficient balancing on the same machine concludes this part of the example. In the present case, the requirement is to zero the relative vibration displacement between disc 1 and the stator at particular speeds. Using only one balance plane, this can be zeroed at only one speed. With two balance planes, this can be zeroed at two speeds and so forth. Balance planes were utilized in numerical order. Figure 8 shows the outcomes. Sharp antiresonances occur in the response at the speeds specified. The general trend is that as more target speeds are chosen (and therefore more planes in this case), the overall response at all frequencies before the first resonance tends to improve. However, in one case this did not happen.

This part of the example is concluded by considering the influence of coefficient balancing on the same machine. In fact, it is common to utilize more target outputs than balance degrees of freedom. If a large number of individual speeds had been chosen between 10 and 100 Hz, the changes occurring with each increase in the number of balance planes used would have been more uniformly good.

5 ROBUST BALANCING, WITH KNOWN PARAMETER DISTRIBUTIONS

In describing the method of influence coefficients, it was noted that numerous different target speeds could be set simultaneously. The definitions of $y'$ and $A'$ in equation (9) facilitate the minimization of a cost function, $J'$, which comprises a weighted sum of cost functions, $J(\Omega)$, for the various different values of $\Omega$. This overall cost function, $J'$, can be expressed as

$$J' = \sum_{j=1}^{Q} J(\Omega_j) = \sum_{j=1}^{Q} q^H R^T A_j^2 R q$$  \hspace{1cm} (14)
where $A_j = A(\Omega_j)$. Substituting for $q$ using equation (1), eliminating $e$ using equation (3) and recognizing that the state of unbalance and bend of a rotor does not depend on speed leads to

$$J' = \sum_{j=1}^{Q} \left\{ z(\Omega_j) + D(\Omega_j)d \right\}^H \left\{ z(\Omega_j) + D(\Omega_j)d \right\} = d^H E d + (d^H g + g^H d) + h$$  \hspace{1cm} (15)

where $E$ is a complex positive-definite Hermitian $(M \times M)$ matrix and $g$ is a complex $M$-dimensional vector. The structure of equation (15) is such that $J'$ is always real and positive. The definitions of $E$, $g$ and $h$ follow from equation (15) as

$$E = \sum_{j=1}^{Q} (D(\Omega_j))^H (D(\Omega_j))$$

$$g = \sum_{j=1}^{Q} (z(\Omega_j))^H (z(\Omega_j))$$

$$h = \sum_{j=1}^{Q} (z(\Omega_j))^H (z(\Omega_j))$$

and their interpretations are clear when it is recognized that they represent (for the uncorrected rotor) the second, first and zeroth derivatives of $J'$ with respect to unbalance corrections in the vector $d$. Since $J'$ is quadratic by definition, all higher derivatives are implicitly zero. The optimal set of corrective unbalance is found to be

$$d_{\text{optimal}} = -E^{-1}g$$  \hspace{1cm} (17)

The preceding discussion has effectively retraced long-past steps [16], but this may now be extended. Shaft speed is one parameter that strongly affects the response of a system to applied forcing from unbalance and rotor bends. However, other parameters also affect this response. The definition of an overall balancing cost function, developed above for the case of multiple shaft speeds, can also be extended to multiple values of other parameters. Suppose that there are $R$ uncertain parameters, $\{\theta_1, \theta_2, \ldots, \theta_R\}$, in the present model for the rotating machine, of which one may be shaft speed. Let the distribution of the general $i$th parameter, $\theta_i$, be represented by a finite number, $S_i$, of values in the set $\Theta_i$. Each one of the possible values allowed for $\theta_i$ has an associated set of weightings, $p_i(\theta_i)$. For any one parameter, $\theta_i$, the sum of the weighting factors $p_i(\theta_i)$ for all $\theta_i \in \Theta_i$ is unity.

In the case where there are three variable parameters ($R = 3$), the overall balancing cost function, $J'$, can be expressed as

$$J' = \sum_{\theta_1 \in \Theta_1} \sum_{\theta_2 \in \Theta_2} \sum_{\theta_3 \in \Theta_3} J(\theta_1, \theta_2, \theta_3)p_1(\theta_1)p_2(\theta_2)p_3(\theta_3)$$  \hspace{1cm} (18)

The extension to higher numbers of variable parameters is obvious. Moreover, by allowing for a continuous range of parameter values in each range, the summations become integrals.

The overall cost function remains a matrix quadratic function of the unbalance correction vector $d$ having the form of equation (15). Hence an optimum balance condition can be determined using equation (17). The weightings $p_i(\theta_i)$ might typically be the probabilities that

Fig. 8  Response (disc 1 relative to the stator) after fixed speed balancing.
parameter $\theta_i$ would have a given value. Note that these are quite different from the weighting (set using matrix $A$) that indicates the relative importance of various synchronous vibration components.

6 ROBUST BALANCING, WITH UNKNOWN PARAMETER DISTRIBUTIONS

In many cases, there is reasonably reliable knowledge that particular parameters lie within some closed range of values, but there is very little real information about the distribution of the parameters within their respective ranges. A pragmatic view might be taken whereby some distribution for each of the parameters could simply be assumed. However, in these cases it is often more sensible to determine an unbalance correction that will minimize the worst possible cost if all of the parameters conspired to have the most unfortunate values. The cost function of interest is still given by equation (15), which must now be regarded as a function of the parameters $J' = J'(\theta_1, \ldots, \theta_R)$. The unbalance correction must be constant; however, $E$, $g$ and $h$ are obtained from equation (14) and all depend on the uncertain parameters $\{\theta_1, \theta_2, \ldots, \theta_R\}$. The problem of minimizing the worst possible cost function for any set of parameters may be expressed as

$$\min_{\mathbf{d}} \max_{\theta_i \in \Theta_i} \{J'(\theta_1, \ldots, \theta_R) : \theta_i \in \Theta_i, i = 1, \ldots, R\}$$

(19)

where vector $\mathbf{d}$ is chosen such that the worst case value of the cost function is as small as possible. For each unbalance $\mathbf{d}$, there will be a range of responses due to the variation in the uncertain parameters, and a particular set of parameters will give a worse case response. The possible values of the unbalance $\mathbf{d}$ are searched and the unbalance that gives the smallest worse case response is chosen.

The closed-form solution of this optimization problem is very difficult. One possibility is to use robust estimation [17, 18], which considers the underlying system of linear equations whose minimum norm gives the cost function [equation (15)]. The parameter uncertainty causes the coefficient matrix and residual vector to change. This parameter uncertainty is encapsulated by defining nominal values of the matrix and vector and then specifying the maximum norm of the difference from these nominal values for all the parameters values. Although this new problem does have a closed-form solution that is related to Tikhonov regularization [19–21], often the redefinition of the uncertainty gives a conservative solution.

The alternative is the direct optimization of equation (19). This is relatively straightforward if the parameters take on discrete values, and so for a given unbalance correction the different sets of parameters may be searched for the worse case cost. For continuous parameter distributions, e.g. if a parameter is within a certain interval, one possibility is to consider a discrete set of parameters spread throughout the range. In many cases such an approach is not necessary, since the worse case cost function will occur at vertices in the parameter space. This gives the exact solution if the parameter changes cause monotonic changes in the cost function, an assumption that is valid in many instances.

The discussion in the next section on distributions of significant residual unbalance concentrates on known parameter uncertainty. Here the matrix $E$ is obtained as a weighted sum over the parameters, using the known parameter weighting. The same approach may be used for unknown parameter uncertainty, where the matrix $E$ is obtained from the worse case response.

7 DISTRIBUTIONS OF SIGNIFICANT RESIDUAL UNBALANCE

In Sections 5 and 6 above, a method was described whereby an overall balancing cost function, $J'$, can be derived encapsulating knowledge about the relative importance of different components of synchronous vibration as well as the spread of model parameters that are known to be uncertain. The method of determining the optimal unbalance correction given an initial state of unbalance (and bend) is straightforward from this point.

This overall balancing cost function, $J'$, is defined by the terms $E$, $g$ and $h$ according to equations (14) and (15) in which $E$ is an $(M \times M)$ Hermitian matrix, $g$ is an $M$-dimensional vector and $h$ is a scalar. Matrix $E$ is obtained from the application of a numerical model of the assembled machine for each possible independent combination of the parameter values, $\theta_i$. This matrix is independent of the initial state of rotor unbalance and bend.

Matrix $E$ contains some extremely useful information. If there is no damping in the model and no gyroscopic effects, the imaginary components within $E$ will be negligible. Singular value decomposition of $E$ leads to

$$E = F^H \Gamma F$$

where $F^H F = I$ (20)

where $\Gamma$ is a real $(M \times M)$ diagonal matrix. In general, $F$ is a complex $(M \times M)$ matrix, but in practice, it is predominantly real.

The results of equation (20) are extremely important. They indicate directly the significance of different independent distributions of corrective unbalance. If some of the diagonal entries of $\Gamma$ are very small, compared with others, the indication is that more unbalance correction degrees of freedom have been provided than are truly useful, which can thus indicate how many balance planes would be useful. This cannot be an absolute conclusion unless some diagonal entries
are truly zero, but the exceptions are so few that they do not warrant serious consideration at this stage. More can be said. If the singular values in $\mathbf{\Gamma}$ are ordered according to size, beginning with the largest, and if $\mathbf{\Gamma}(j,j) \leq \mathbf{\Gamma}(L,L)$ for all $j > L$, then the indication is that only $L$ of the original $M$ balance planes really needs to be used. An optimal (or near-optimal) subset of $L$ balancing planes can be chosen from the original set of $M$ planes provided using a straightforward extension of the method of effective independence, which is exactly what has been applied to the problem of determining near-optimal sets of measurement locations for modal tests on structures [22].

There is one final positive outcome from the singular value decomposition results that will be outlined here but expanded in the following section. Knowing that there are only $L$ significant distributions of residual unbalance on the rotor and knowing their shapes provides for the development of dedicated unbalance detection tests such that the requisite unbalance information can be picked out easily. The actual state of residual unbalance on a rotor is not relevant. What is relevant is the projection of this actual state of residual unbalance on to the much smaller subspace defined by the $L$ vectors of significant residual unbalance.

8 THE SIMULATED EXAMPLE REVISITED

The same example system is examined again. Figure 9 shows the same system again but indicates that there is uncertainty about one of the point masses on the stator and that one of the springs connecting the rotor to the stator is also uncertain. Table 2 describes crudely the probability density function that describes the possible values of the uncertain point mass and Table 3 indicates in a similar way the spread of values expected for the uncertain spring along with the respective probabilities. A rough interpretation of the information in Table 2 would be that the probability that the uncertain mass will lie in the range $300 \times 10^{-0.05}$ to $300 \times 10^{-0.05}$ kg is around $0.7/0.7$ (note that $0.6 + 0.7 + 0.8 + 0.9 + 1.0 + 0.9 + 0.8 + 0.7 + 0.6 = 7$) whereas the probability that the uncertain mass will lie in the range $300 \times 10^{-0.35}$ to $300 \times 10^{-0.25}$ kg is $0.7/0.7 = 0.1$.

The vector of synchronous outputs of importance now comprises three entries:

(a) displacement of disc 1 on the rotor relative to the stator,
(b) absolute displacement of the stator at the end of the first stator element from the left-hand end and
(c) absolute displacement of disc 2 of the rotor.

The relative importance of these three synchronous vibration components is reflected in the following $\mathbf{A}$ matrix:

$$
\mathbf{A} = \begin{bmatrix}
1.5 & 0 & 0 \\
0 & 2.0 & 0 \\
0 & 0 & 1.0
\end{bmatrix}
$$

The matrix $\mathbf{E}$ was assembled according to the logic outlined above with nine values for each of the two parameters and one hundred values for shaft speed in geometric spacing from 20 Hz up to 200 Hz. In order to ensure that $\mathbf{E}$ is not dominated by one particular shaft speed and one particular combination of parameters (which could happen if one of the speeds sampled happened to be very close to a resonance), a notional damping factor of 1 per cent of critical damping is enforced on each of the modes. Table 4 shows the optimal balance corrections that were computed by only allowing balance planes BP1 to BP3 to be corrected. The modal damping has caused the unbalance corrections to be complex.

![Fig. 9 Uncertain quantities in the simulated system](image-url)
Suppose instead that the possible parameter values are assumed to lie at the limits of the ranges given in Tables 2 and 3, so that, for example, the mass lies somewhere between $300 \times 10^{-3}$ to $300 \times 10^{-6}$ kg. Direct optimization of equation (19) is performed, where the discrete parameter values in Tables 2 and 3 are used, and the worst cost value taken. The resulting unbalance correction is shown in Table 4 and corresponds to the vertices in the parameter space corresponding to a mass of $300 \times 10^{-3}$ kg and a stiffness of either $10^7$ or $10^8$ N/m. The balance correction in this case is quite similar to that obtained from the known parameter uncertainty.

For comparison, Table 4 also gives the unbalance correction obtained using a fixed set of parameters, with a mass of 300 kg and a stiffness of 100 MN/m. The various cost functions are used to assess the performance of the three unbalance corrections, and these are also shown in Table 4, as well as the values of the cost functions before balancing. The cost function values are consistent, since, for example, if the parameter distribution is known, then equation (15) is minimized and the unbalance correction obtained by assuming an unknown balance distribution must be higher. The advantage of including the parameter uncertainty is highlighted by the poor performance of unbalance correction using a fixed set of parameters.

Table 2  The variation in the mass parameters

<table>
<thead>
<tr>
<th>Mass/300 kg</th>
<th>$10^{-3}$</th>
<th>$10^{-2}$</th>
<th>$10^{-1}$</th>
<th>$10^{0}$</th>
<th>$10^{1}$</th>
<th>$10^{2}$</th>
<th>$10^{3}$</th>
<th>$10^{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_0$/10 kg m$^2$</td>
<td>$10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$10^{0}$</td>
<td>$10^{1}$</td>
<td>$10^{2}$</td>
<td>$10^{3}$</td>
<td>$10^{4}$</td>
</tr>
<tr>
<td>Relative probability</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 3  The variation in the stiffness parameters

<table>
<thead>
<tr>
<th>$K_d$(N/m)</th>
<th>$10^{3}$</th>
<th>$10^{4}$</th>
<th>$10^{5}$</th>
<th>$10^{6}$</th>
<th>$10^{7}$</th>
<th>$10^{8}$</th>
<th>$10^{9}$</th>
<th>$10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative probability</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 4  Balance corrections with parameter uncertainty

<table>
<thead>
<tr>
<th></th>
<th>Original unbalance</th>
<th>Fixed parameters: $J_0 = 10$kg m$^2$, $K_d = 100$MN/m</th>
<th>Distribution of parameter uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Known (Section 5)</td>
<td>Unknown (Section 6)</td>
<td></td>
</tr>
<tr>
<td>Correction at BP 1 (g m)</td>
<td>$0.7181 + 0.000535 j$</td>
<td>$2.041 + 0.00102 j$</td>
<td></td>
</tr>
<tr>
<td>Correction at BP 2 (g m)</td>
<td>$2.062 - 0.00314 j$</td>
<td>$1.515 - 0.00015 j$</td>
<td></td>
</tr>
<tr>
<td>Correction at BP 3 (g m)</td>
<td>$-1.088 - 0.00419 j$</td>
<td>$-2.219 - 0.00180 j$</td>
<td></td>
</tr>
<tr>
<td>Cost [equation (15)]</td>
<td>$0.8087$</td>
<td>$3.906 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>Cost [equation (20)]</td>
<td>$0.9621$</td>
<td>$7.088 \times 10^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>

9 MEASURING DISTRIBUTIONS OF RESIDUAL UNBALANCE

The central message of this paper is that the optimal unbalance corrections applying to a given rotor are not, in general, those that would be determined using standard balancing methods. Knowledge of the exact residual distribution of unbalance on a rotor coupled with knowledge of the dynamic characteristics of the rotor itself, the stator in which it will be deployed and the important measures of unbalance originated vibration can be combined to determine the optimal unbalance correction in any given situation.

An apparent weakness in the above is the implied requirement that the exact distribution of residual unbalance of a rotor must be determined. In fact, it is generally impossible to determine this distribution exactly with a finite amount of effort. What can be determined relatively easily are the projections of the distribution of residual unbalance on to the natural modes of the rotor in the balancing machine (or on to the free-free modes of the rotor if bearing reaction forces are being measured) for the low-frequency modes. If the intention is to cancel out the net unbalance...
excitation of these modes directly, then the measurement task is relatively straightforward.

Since the procedure advocated in this paper does not simply involve zeroing the net excitation of the low-frequency modes of the rotor (either free–free or in the balancing machine), the task is more complex. In this case, the requirement is to determine the projection of the actual distribution of residual unbalance on to the characteristic vectors of significant residual unbalance, found from the singular value decomposition of equation (20). This requires that the information obtained from a normal balancing run should be enriched. It is beyond the scope of this paper to discuss all of the possible ways in which this could be achieved, but some possibilities are:

1. Measuring static (in the rotor frame) strains on the rotor as well as deflections, since this measurement will contain significant contributions from the higher modes. Rigid-body modes do not contribute to straining.

2. Perturb boundary conditions by fixing auxiliary systems on to the ends of the rotor such that the combined rotor and auxiliary systems have natural modes very similar (on the rotor) to the distributions of significant residual unbalance.

3. Assembled rotors provide the information directly, given 'line-segment' approximations of the distribution of residual unbalance for each individual rotor component.

10 CONCLUSIONS

This paper has considered the effect of variability within the foundations of a rotating machine on the optimum balance corrections. The parameter variability may be given in terms of a probability distribution or in terms of fixed limits (convex sets). Unbalance corrections have been derived in both cases. Of fundamental importance is the concept of distributions of significant residual unbalance, which determines the optimum balance planes for a given machine and parameter variability, and plays a similar role to the machine modes in modal balancing. Robust balancing has applications to machines that are balanced in a balancing machine, where the stators are different, and also to machines whose foundation characteristics change with time. An example of the latter case is a machine mounted on elastomer supports, whose stiffness and damping changes with temperature and also due to ageing of the material.

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REFERENCES


