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MODEL UPDATING USING ROBUST ESTIMATION

C. MARES

Department of Engineering (Mechanical Engineering Division), University of Liverpool, Brownlow Hill, Liverpool L69 3GH, U.K.

M. I. FRISWELL

Department of Mechanical Engineering, University of Wales Swansea, Swansea SA2 8PP, U.K.

AND

J. E. MOTTERSHEAD

Department of Engineering (Mechanical Engineering Division), University of Liverpool, Brownlow Hill, Liverpool L69 3GH, U.K.

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Model updating attempts to correct errors in a finite element model using measured data. However, the measurements are corrupted with noise and the finite element model contains errors. Most updating schemes try to find the best model in some mean sense. This paper takes a different view and uses robust estimation techniques to determine the most robust parameter set. This parameter set is optimum in that the residuals are as small as can be, for a range of bounded uncertainties on the model and measurements. The relationship between this robust identification and Tikhonov regularisation is explored. The use of the 'L' curve method and expected parameter uncertainty to determine the regularisation parameter are shown. The approach is demonstrated using a simulated cantilever beam example, and experimental data from the GARTEUR test structure.

1. INTRODUCTION

Finite element model updating has become a viable approach to increase the correlation between the dynamic response of a structure and the predictions from a model. In model updating parameters of the model are adjusted to reduce a penalty function based on residuals between a measurement set and the corresponding model predictions. Typical measurements include the modal model (natural frequencies and mode shapes) and frequency response functions. The choice of penalty function, and also the optimisation approach, has been the subject of considerable research and are well covered by the authors' survey paper, book and recent special issue of Mechanical Systems and Signal Processing [1–3]. The treatment of the noise in the measurements and the errors in the finite element model is critical to obtain good estimates of the uncertain parameters. Often these errors are estimated in terms of variances where the statistical distributions associated with the parameters and measurements are assumed to be Gaussian. Once these probability distributions are specified the full power of probability theory may be brought to bear. However, there is often insufficient information to even specify the variance, let alone determine the probability distribution.

Methods that specify uncertainties as bounds or convex sets have been popular in control engineering for some time and are beginning to appear in structural dynamics [4–8]. This paper is concerned with updating models of structures, where the errors are specified as

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bounds. One approach to the estimation problem is to determine the bounds on the uncertain parameters, leading to the calculation of the feasible set of parameters which invariably has complex boundaries [6, 7]. The approach adopted here is the estimation of a single set of parameters, which is optimal in a min–max sense to be defined later [9–11].

2. BOUNDED DATA UNCERTAINTIES

The approach of Chandrasekaran *et al.* [9] will now be outlined. Much of their paper is concerned with the proof of a theorem, which is stated below. Suppose the updating problem has been linearised so that the solution is sought to the linear equation

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{b} \in \Re^m$ contains the error in the measurements, $\mathbf{A} \in \Re^{m \times n}$ is the sensitivity matrix and $\mathbf{x} \in \Re^n$ the vector of parameter changes [2]. Equation (1) has m measurements, which produce the observation vector, and n unknown parameters. It is assumed that $m \geqslant n$, that is the problem is over-determined and there are more measurements than parameters. Although the quantities in equation (1) are assumed real, complex equations may be considered by separating the real and imaginary parts.

In equation (1) the sensitivity matrix, $\mathbf{A}$, and observation vector, $\mathbf{b}$, are assumed to be known. In fact these quantities are subject to errors, and the true coefficient matrix is $\mathbf{A} + \delta\mathbf{A}$, and the true observation vector is $\mathbf{b} + \delta\mathbf{b}$. $\delta\mathbf{A}$ and $\delta\mathbf{b}$ arise from modelling errors and parameter uncertainty and $\delta\mathbf{b}$ also arises from measurement noise. We suppose that the upper bounds, η and η_b, of these errors are given, so that

$$\|\delta\mathbf{A}\|_2 \leqslant \eta \tag{2}$$

and

$$\|\delta\mathbf{b}\|_2 \leqslant \eta_b. \tag{3}$$

We can now state the problem as follows: Given non-negative η and η_b determine the best parameter estimate $\hat{\mathbf{x}}$ as the one that solves the following min–max problem:

$$\min_{\hat{\mathbf{x}}} \max \{\|(\mathbf{A} + \delta\mathbf{A})\hat{\mathbf{x}} - (\mathbf{b} + \delta\mathbf{b})\|_2 : \|\delta\mathbf{A}\|_2 \leqslant \eta,\ \|\delta\mathbf{b}_2\| \leqslant \eta_b\}. \tag{4}$$

The significance of equation (4) is not readily apparent and some more detailed interpretation will now be given. For any given $\hat{\mathbf{x}}$, there are many residuals,

$$\|(\mathbf{A} + \delta\mathbf{A})\hat{\mathbf{x}} - (\mathbf{b} + \delta\mathbf{b})_2\| \tag{5}$$

as each feasible $\delta\mathbf{A}$ and $\delta\mathbf{b}$ will give a different residual. For a given $\hat{\mathbf{x}}$ there will be a maximum value of these residuals. A second choice of $\hat{\mathbf{x}}$ will produce a different maximum residual. We want to choose the parameter estimate $\hat{\mathbf{x}}$ that produces the smallest maximum residual. Thus, for all possible errors in the coefficient matrix and residual vector we have the smallest maximum possible error. The physical interpretation of this is that the residual is the smallest one of all the cases of most adverse uncertainty. Chandrasekaran *et al.* [9] gave a geometric interpretation of this optimisation. Of course if $\eta = \eta_b = 0$ then the min–max problem reduces to the standard least-squares problem. El Ghaoui and Lebret [11] considered the same problem, but derived a different method for its solution. As it turns out, the solution to equation (4) is independent of η_b. This may be understood by using the triangle inequality for vector norms to give

$$\|(\mathbf{A} + \delta\mathbf{A})\hat{\mathbf{x}} - (\mathbf{b} + \delta\mathbf{b})\|_2 \leqslant \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2 + \eta\|\hat{\mathbf{x}}\|_2 + \eta_b. \tag{6}$$

Chandrasekaran *et al.* [9] showed that the min–max problem, equation (4), always has a solution if **A** is full rank, and gave a solution method. The solution is constructed as follows:

(1) Compute the singular-value decomposition of **A**,

$$\mathbf{A} = \mathbf{U}\begin{bmatrix}\Sigma \\ \mathbf{0}\end{bmatrix}\mathbf{V}^T \tag{7}$$

where **U** and **V** are orthogonal, and $\Sigma = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with the singular values arranged in descending order

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0 \tag{8}$$

(2) Partition the vector $\mathbf{U}^T\mathbf{b}$ into

$$\begin{Bmatrix}\mathbf{b}_1 \\ \mathbf{b}_2\end{Bmatrix} = \mathbf{U}^T\mathbf{b} \tag{9}$$

where $\mathbf{b}_1 \in \Re^n$ and $\mathbf{b}_2 \in \Re^{m-n}$.

(3) It is useful to introduce the following function:

$$G(\alpha) = \mathbf{b}_1^T(\Sigma^2 - \eta^2\mathbf{I})(\Sigma^2 + \alpha\mathbf{I})^{-2}\mathbf{b}_1 - \frac{\eta^2}{\alpha^2}\|\mathbf{b}_2\|_2^2. \tag{10}$$

Chandrasekaran *et al.* [9] proved that $G(\alpha) = 0$ has a unique positive root. This root will feature in the parameter solutions.

(4) Define

$$\tau_1 = \frac{\|\Sigma^{-1}\mathbf{b}_1\|_2}{\|\Sigma^{-2}\mathbf{b}_1\|_2} \tag{11}$$

and

$$\tau_2 = \frac{\|\mathbf{A}^T\mathbf{b}\|_2}{\|\mathbf{b}\|_2}. \tag{12}$$

The form of the solution will depend on the magnitude of the coefficient matrix uncertainty, η, compared to τ_1 and τ_2, and also on whether the residual vector **b** is in the column space of **A**.

Case 1. If $\eta \geqslant \tau_2$ then $\hat{\mathbf{x}} = \mathbf{0}$.

Case 2. If $\eta < \tau_2$ and **b** is not in the column space of **A**, then

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A} + \hat{\alpha}\mathbf{I})^{-1}\mathbf{A}^T\mathbf{b} \tag{13}$$

where $\hat{\alpha}$ is solution of $G(\hat{\alpha}) = 0$. This is the most common case, and the result is identical to Tikhonov regularisation [12–14], where now the regularisation parameter is determined from the uncertainty η.

Case 3. If **b** is in the column space of **A**, then equation (13) is the solution only if $\tau_1 < \eta < \tau_2$. If $\eta < \tau_1$ then the solution is the standard least-squares solution, and if $\eta = \tau_1 = \tau_2$ there are infinitely many solutions.

3. TIKHONOV REGULARISATION

The solution given by equation (13) is essentially a Tikhonov regularisation where the regularisation parameter is obtained from the uncertainty in **A** given by η. Tikhonov regularisation minimises

$$J(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \alpha\|\mathbf{x}\|_2^2 \tag{14}$$

where the parameter α is either given or must be determined. Notice that the essential difference between Tikhonov regularisation and equation (4) is the use of squared norms as opposed to norms, as highlighted by equation (6).

One useful approach to determining α, when the bounds η and η_b are not available, is via the 'L' curve method of Hansen [15]. For a given value of α equation (14) can be minimised to give the optimum estimate of the parameters as

$$\mathbf{x}_\alpha = [\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I}]^{-1}\mathbf{A}^T\mathbf{b}. \tag{15}$$

Notice that the estimated parameters depend on α. The 'L' curve is a plot of the norm of the side constraint, $\|\mathbf{x}_\alpha\|$, against the norm of the residual, $\|\mathbf{Ax}_\alpha - \mathbf{b}\|$, for different values of α. Hansen [15] showed that the norm of the side constraint is a monotonically decreasing function of the norm of the residual. He pointed out that for a reasonable signal-to-noise ratio and the satisfaction of the Picard condition, the curve is approximately vertical for $\alpha < \alpha_{\text{opt}}$, and soon becomes a horizontal line when $\alpha > \alpha_{\text{opt}}$, with a corner near the optimal regularisation parameter α_{opt}. The approach is called the 'L' curve method because of the shape of this curve. The optimum value of the regularisation parameter, α_{opt}, corresponds to the point with maximum curvature at the corner of the log–log plot of the 'L' curve. This point represents a balance between confidence in the measurements and the analyst's intuition.

If the bounds η and η_b are available then Prells [16, 17] suggested an approach to obtain the optimum value of the regularisation parameter α. He showed that α is a solution of

$$\mathbf{b}^T[\mathbf{AA}^T + \alpha\mathbf{I}]^{-1}[\alpha^2\mathbf{I} - \eta^2\mathbf{AA}^T][\mathbf{AA}^T + \alpha\mathbf{I}]^{-1}\mathbf{b} = \eta_b^2. \tag{16}$$

Notice that in this approach, the upper bound on the measurement errors, η_b, also influences the optimum value of α.

4. THE APPLICATION TO MODEL UPDATING

The application of the robust estimation techniques to model updating will be demonstrated using eigenvalue residuals. Other residuals could be used [18] and although the parameters estimated would be different the approach to the regularisation would be equivalent. Suppose a finite element model of the structure exists and the parameters to be estimated using model updating have been selected. The purpose of model updating is to minimise [2]

$$J(\mathbf{y}) = \sum_{i=1}^{n} \|\lambda_i(\mathbf{y}) - \lambda_{e,i}\|^2 \tag{17}$$

where $\lambda_{e,i}$ is the ith measured eigenvalue and λ_i is the ith eigenvalue predicted by the finite element model (and therefore depends on the parameter values). $\mathbf{y}$ is the vector of parameters to be updated.

The optimisation problem given by equation (17) is non-linear when eigenvalue residuals are used. The standard approach adopted in model updating is to linearise the estimation problem about the current parameter estimate, and to iterate until convergence [2]. There are a number of issues relating to the conditioning of the estimation problem that must be addressed.

Weighting is applied to both the measured data and the parameters. Generally, the weighting on the measured data is such that the entries in the eigenvalue residuals are divided by the corresponding measured eigenvalue. Furthermore, the unknown parameters are usually normalised so that the initial value of these parameters is unity. Using this approach to the weighting of the measurements and the parameters leads to non-dimensional quantities that help with the scaling of the 'L' curve. However, increased weight may be applied to eigenvalues that are accurately measured and to parameters that are not expected to change significantly from their initial values.

Often the number of uncertain parameters is large compared to the number of measurements, resulting in an under-determined estimation problem. Even if there are less parameters than measurements the resulting estimation problem can be ill-conditioned. In this case extra information must be provided to produce a well-conditioned estimation problem. The proper choice of these constraints is vital to ensure that the updated parameters have physical meaning. This has be considered at length elsewhere [12, 19] and is not considered further in this paper. Here we only consider the constraint that the parameters deviate as little as possible from their initial values. This constraint and the linearised form of equation (17) are easily transformed to the general form of equation (1).

Of course since model updating is iterative, the robust estimation procedure must be applied at each iteration. In robust updating the value of η is fixed at the outset (or perhaps at the first iteration), and thus α changes at each iteration. In Tikhonov regularisation either α is fixed, or else α is estimated at each iteration by the 'L' curve method. However in practice, with a suitable choice of parameters, convergence is fast and the change in these regularisation parameters is small.

The choice of η is crucial to successful robust updating. One approach is to determine the expected bounds directly by assuming that the unknown parameter vector lies in a given convex set. For any particular parameter vector within this set, the matrix $\mathbf{A}$ is computed. The norm of the difference between this matrix, and that obtained using the initial parameter vector is obtained. This is repeated for all parameter vectors within the set and the maximum found, which produces a numerical estimate of η. If the relationship between $\mathbf{A}$ and the parameters is well behaved, then it is often sufficient to look at the vertices in the parameter space. This direct approach has some difficulty since the errors depend not only on the parameter errors but also on model structure errors. For Tikhonov regularisation, the 'L' curve method is able to estimate a reasonable value of the regularisation parameter. A similar approach for robust updating is desirable. One possibility is to find the corner of the 'L' curve at the first iteration, and use the roots of $G(\alpha) = 0$, now considered as a function η, to find the corresponding value of η. Friswell *et al.* [20] estimated η using the total least-squares approach suggested by El Ghaoui and Lebret [11]. Unfortunately, this gave consistently low values of η and the results were very close to those obtained by standard least squares.

No mention has been made about implementing these procedures for large-scale models with many degrees of freedom. The great advantage of using eigenvalue residuals is that to a large extent the size of the finite element model is immaterial. The conditioning of the estimation problem, and the quality of updated model, is totally dependent on the information content of the measured data used, and the correct choice of parameters. The parameters should be chosen so that their effect on the predictions of the measured data vector are as linearly independent as possible. With large models of complex structures, where there are a huge number of candidate parameters, this choice is not easy. However, the choice of parameters is beyond the scope of this paper, and will not be consider further.

5. A SIMULATED CANTILEVER BEAM EXAMPLE

The robust updating approach was tested on a simulated cantilever beam example. A simulated example has the advantage that the expected answer is known, and the identification can be performed for a number of nominal systems with bounded uncertainty. Figure 1 shows the simulated beam with a discrete spring and mass. The beam is 1 m long, 2.5 cm thick and 5 cm wide and the material properties of the beam are those of steel. The beam is simulated using 10 elements, and five eigenvalues are assumed to be measured. The parameters are as follows

$$y_1 = k/k_0, \qquad y_2 = m/m_0, \qquad y_3 = EI/EI_0 \tag{18}$$

where $k_0 = 4$ kN/m, $m_0 = 0.2$ kg and $EI_0 = 13.7\ \text{kN m}^2$ are the simulated spring stiffness, discrete mass and beam flexural rigidity, respectively. All identification exercises start with parameter values of 0.9, and clearly the 'true' values of all parameters is 1. However, noise is added to the 'measurements', so it cannot be expected that the identified parameters will equal unity. The random noise is from a uniform distribution, which may be positive or negative, with a maximum amplitude of 1% of the eigenvalue.

For this example η is estimated in two ways. First the three parameters are varied simultaneously in the range 0.9–1.1, and for each set of parameter values the matrix **A** is computed. η is then given by the maximum, over all parameter sets, of the difference between this matrix and **A** evaluated when all the parameters are unity. Using this approach $\eta = 0.02303$. This estimated η is possible because we have some idea of the variation in the parameters. An alternative, as explained above, is to find corner of the 'L' curve and then compute the corresponding value of η.

Figure 2 shows the 'L' curve for this example for a typical noise sample, based on Tikhonov regularisation, equation (14), for a range of values of α. This 'L' curve is for the first iteration of the updating approach, and similar curves could be obtained at subsequent iterations. The corner of the 'L' curve, and also the points corresponding to the robust updating approach with $\eta = 0.02303$ and the least-squares approach (equivalent to $\alpha = 0$), are marked. Clearly, the robust estimation places slightly less emphasis on the initial parameter values than the regularised solution at the corner of the 'L' curve. However, the point corresponding to the robust estimation is quite close to the corner.

The alternative is to calculate the value of α directly, based on the upper bounds η and η_b, by solving equation (16). Of course, we now need to estimate η_b, although for the simulation this may be approximated as 1% of the norm of **b** since a known level of noise has been added. This approach gives $\eta_b = 0.002093$. Table 1 shows the values of α obtained for different values of η and η_b. Clearly, with the estimated values of η and η_b, α is very small and the solution is close to the least-squares solution. This situation arises because the assumed error bounds are small. As the error bounds increase then α changes significantly. For

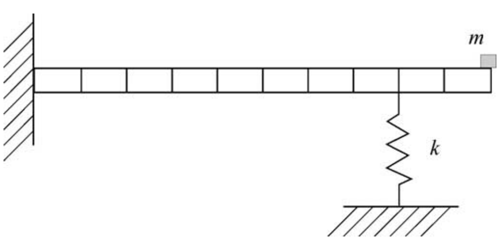

Figure 1. The cantilever beam example.

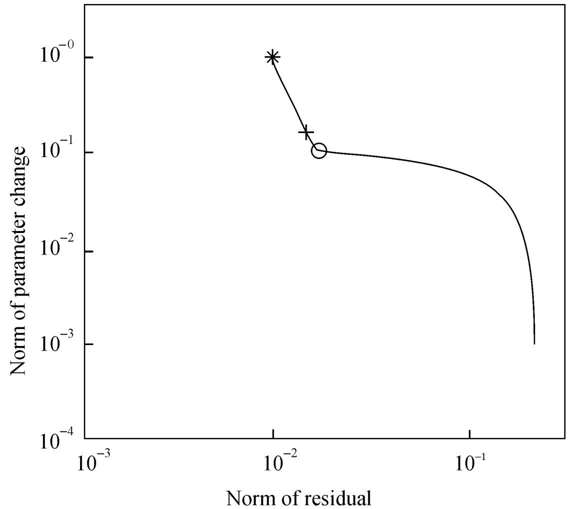

Figure 2. The 'L' curve for the cantilever beam example. —, L curve; ○, Corner of the L curve; +, η from parameter changes; $*$, Least squares solution.

TABLE 1

The calculated values of α using equation (16) *for different values of η and η_b ($\eta_0 = 0.02303$ and $\eta_{0b} = 0.002093$)*

	η_b			
η	η_{0b}	$2\eta_{0b}$	$5\eta_{0b}$	$10\eta_{0b}$
η_0	0.000185	0.000200	0.000409	0.29812
$2\eta_0$	0.000514	0.000547	0.000984	0.31183
$5\eta_0$	0.002931	0.003145	0.006362	0.39424
$10\eta_0$	0.27158	0.28490	0.36870	0.60073

comparison $\alpha = 0.002505$ for the robust estimation. The sensitivity of α, and therefore the parameter estimates, to the bounds makes it difficult to justify choosing a single α.

The estimation process was run for 200 noise samples using least-squares estimation, robust least-squares estimation with $\eta = 0.02303$ and using η obtained from the corner of the 'L' curve at the first iteration. Table 2 shows the mean and standard deviation of the parameter estimates obtained at convergence. Also shown are the errors in the eigenvalues on convergence. As expected, the higher the value of η, or α, the more weight is placed on the initial parameters. This in turn means that the estimated parameters are closer to the initial parameters, and the errors in the predicted eigenvalues are greater. Thus, in strictly statistical terms, robust updating produces estimated parameters with a lower standard deviation, but increased bias. This bias arises because of the confidence in the initial parameters, and not because the method has performed poorly. Interestingly, parameter 3 is always updated accurately, mainly because the eigenvalues are very sensitive to this parameter. What is clear from the results is that the standard deviation of the parameters estimated from the least-squares solution are much larger than those estimated from robust least squares. In a practical example there will be only one set of measurements, and thus

TABLE 2

The statistics of the identified parameters and eigenvalue errors (in %) for the beam example

	Least squares		Robust least squares			
			Parameter variation		'L' curve	
	Mean	SD	Mean	SD	Mean	SD
k	0.9978	0.3519	0.8942	0.0423	0.8939	0.0073
m	1.0303	0.2096	0.9361	0.0683	0.9050	0.0115
EI	0.9984	0.0201	0.9937	0.0038	0.9937	0.0025
Mode 1	0.0000	0.0006	− 0.1059	0.2043	− 0.2516	0.4553
Mode 2	0.0122	0.2934	0.1741	0.4358	0.1759	0.4532
Mode 3	− 0.0555	0.4707	0.0000	0.4766	0.0008	0.4828
Mode 4	0.0698	0.5021	0.0370	0.4913	0.0373	0.4949
Mode 5	− 0.0335	0.3626	− 0.1381	0.4878	− 0.1379	0.5190

the identified parameters are very likely to be a long way from the mean, and therefore the identified model will be relatively inaccurate.

6. THE GARTEUR SM-AG19 TEST STRUCTURE

The Structures and Materials Action Group (SM-AG19) of the Group for Aeronautical Research and Technology in EURope (GARTEUR) was initiated in 1995 with the purpose of comparing a number of current measurement and identification techniques applied to a common structure [21–23]. The testbed was designed and manufactured by ONERA and has now been accepted as a benchmark structure by the working group on Finite Element Model Updating of the COST Action F3 on Strucural Dynamics [24, 25]. The testbed represents a typical aircraft design with a fuselage, wings and a tail. Realistic damping levels are achieved by the application of a viscoelastic tape bonded to the upper surface of the wings and covered by a thin aluminium constraining layer. The testbed has a wingspan of 2 m and a length of 1.5 m.

The experimental data-base (supplied by the University of Manchester and DLR) consists of 24 frequency response functions for an excitation at the right wing tip in the range 0–80 Hz, 24 frequency response functions for excitations at each of the right and left wing tips in the range 4–65 Hz, and 14 normal modes in the range 6.38–151.32 Hz.

6.1. MODELING ASPECTS

The test structure [21] consists of aluminium beams of rectangular cross-section. The modelling uncertainty, which we aim to reduce by model updating, is mainly concentrated at the joints and, to a lesser extent, at the constrained viscoelastic layer that runs over the length of the wings. In the physical structure the joint between the fuselage and the wing is achieved by screwed connections through a small plate which is sandwiched between the two. In several modes this joint is at a vibration node in which case its representation in the finite element model is not critical. In other modes, however, the joint is strained considerably and the form of the model in the region of the joint may then be important. This joint is the main one that influences the choice of beams or plates for the finite element model. The wings and fuselage are long slender structures which might suggest the use of beam

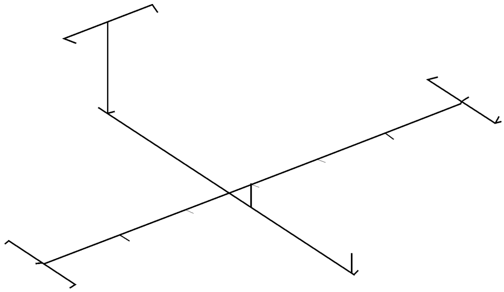

Figure 3. The finite element model of the GARTEUR test structure.

elements. However, they are not connected at the neutral axes. Therefore, locally at the joint there will almost certainly be deformations that cannot be properly represented by the assumption of plane sections remaining plane. The counter argument, for beam-like structures, is that a fine mesh of plate elements would be needed to achieve the same accuracy that can be readily obtained from a small number of Hermitian beams. We choose to construct a finite element model consisting of 126 beams, 26 rigid elements and 5 lumped masses. The complete model, shown in Fig. 3, has 534 degrees of freedom. An equivalent beam model is used to represent the joint between the fuselage and wings. The fuselage imparts an added stiffness to the wings at the joint and similarly the wings have a stiffening effect on the fuselage as they pass over it. Beam offsets are used to model these stiffening effects. The damping and constraining layers on the wings are each represented by beams with nodes offset to the neutral axis of the main wing-beam. Rigid elements are used to connect the winglets at the wing-tips. Offsets are used at the connection between the fin and the tail-plane and between the fin and the fuselage. A point mass (and a point inertia) at the fin/tail-plane joint represents the mass of additional parts used in the construction of the joint.

6.2. THE SELECTION OF UPDATING PARAMETERS

The choice of updating parameters is an important aspect of the finite element model updating process [1, 2]. The sensitive parameters chosen must also be justified by engineering understanding of the test structure [26–30]. The flexural and torsional rigidities (EI_y and GI_t) of the wings are of course very sensitive because the wings are the most active components in all of the lower modes. Their use as updating parameters seems reasonable mainly because of uncertainty in the thickness of the viscoelastic and constraining layers. The beam-offset, $(a_b)_w$, used to stiffen the wings at the joint with the fuselage represents an unrealistic rigid joint and must be adjusted. They are sensitive for the same reason that the wing rigidities are sensitive. At the seventh mode in-plane wing bending occurs with the two wing-tips in anti-phase. This puts the equivalent beam separating the wings and fuselage into torsion and it is sensitive for that mode. The torsional eigenvalue, $(q_2)_{\text{con}}$, of the stiffness matrix (equivalently the torsional rigidity) of the equivalent beam is updated. Two updating parameters at the tail are used. These are the offset, $(a_b)_t$, at the tail-fin/fuselage joint and the first in-plane bending eigenvalue, $(q_1)_t$, of the substructure stiffness matrix for the three elements at the tail-plane/fin joint. Mares *et al.* [31] described the choice of parameters in more detail.

In summary the six parameters used for updating were:

- The first in-plane bending eigenvalue of the substructure stiffness matrix for the three elements at the tail-plane/fin joint, $(q_1)_t$.

- The beam offset at the wing/fuselage joint, $(a_b)_w$.
- The flexural rigidity of the wing, EI_y.
- The torsional rigidity of the wing, GI_t.
- The beam offset at the tail-fin/fuselage joint, $(a_b)_t$.
- The torsional eigenvalue of the stiffness matrix of the beam joining the wing and fuselage, $(q_2)_{con}$.

7. ROBUST UPDATING OF THE GARTEUR STRUCTURE

The six uncertain parameters were updated using the lower nine measured natural frequencies ($f_1 - f_9$). The remaining five natural frequencies ($f_{10} - f_{14}$) were used only for checking the capacity of the updated model to predict natural frequencies outside the frequency range, and thus gauge the quality of the updated model. The measured residuals were weighted with the inverse of the measured natural frequencies, and the parameters normalised so that the initial parameter values were unity. The residuals were weighted further based on the estimated accuracy of the measurements by engineering judgement. Figure 4 shows how the value of α obtained from equation (10) varies with η for the first iteration in this example. As expected a single value of α is obtained for each value of η and α is monotonically increasing. At each value of η, we may obtain the corresponding value of α, and then find the updated parameter values, given by equation (13). These identified parameter values are shown in Fig. 5, which shows the changes from the initial parameter values of unity for the first iteration. As η increases, so does α and thus the initial parameter values are more heavily weighted. This causes the changes from the initial parameter values to be smaller as η increases.

The question of how to choose η then arises. The 'L' curve method is an established approach for choosing or α in Tikhonov regularisation [15]. The resulting α produces the best compromise between the weighting of the residuals and the parameter changes. Figure 6 shows the 'L' curve for the first iteration and shows a corner at about $\alpha = 0.4863$. This would be a good choice of α, which was then converted, by solving $G(\alpha) = 0$, to give $\eta = 0.7137$. Table 3 shows the initial, measured and updated natural frequencies for three different values of η close to this corner. Unfortunately, the equations were too ill-conditioned for the standard least-squares solution to be convergent. However, the results for $\eta = 0.1$ are included, which is close to the smallest η for which the parameters converge.

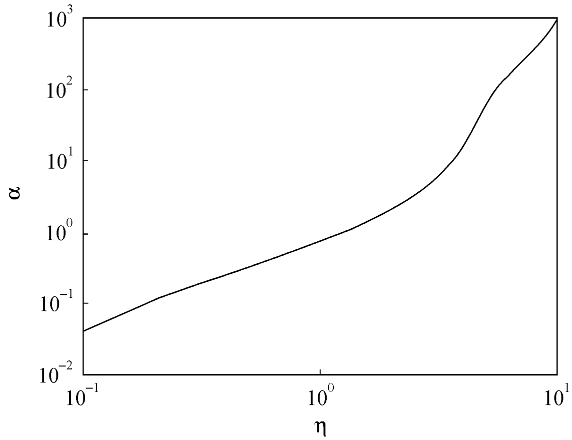

Figure 4. The relationship between η and α for the first iteration.

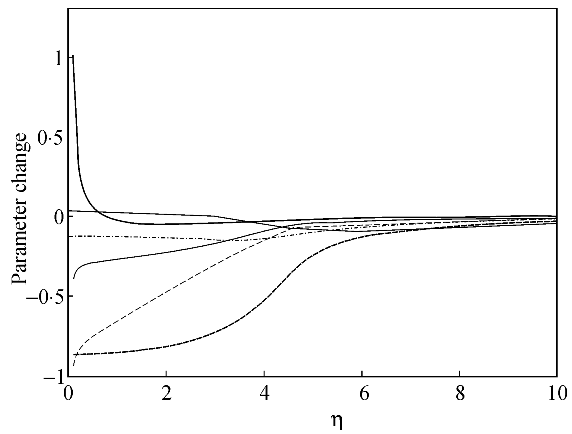

Figure 5. The updated parameters after the first iteration for different values of η. —, $(q_1)_t$; ‒ ‒, $(a_b)_w$; ......, EI_y; . ‒ . ‒, GI_t; —, $(a_b)_t$; ‒ ‒, $(q_2)_{con}$.

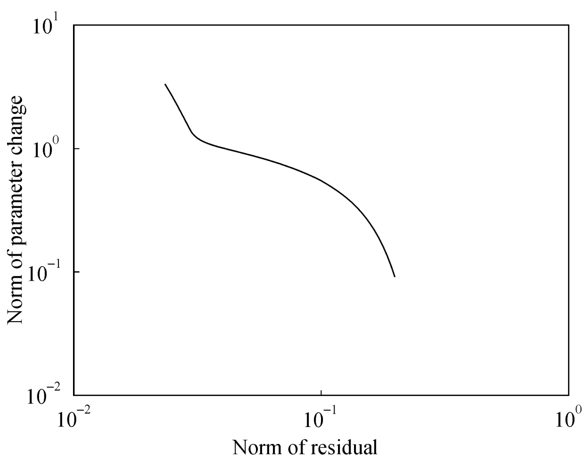

Figure 6. The 'L' curve for the first iteration.

Table 4 gives the corresponding updated parameters. Figure 7 shows the evolution of the estimated parameters for four iterations using a constant value of $\eta = 1$. Convergence was very fast and the updated model not only improves the estimates of the first nine natural frequencies, it also significantly improves the estimates of the five higher natural frequencies not used in the updating. As expected as η increases more weight is given to the initial parameter values, and the changes from their initial values is smaller.

Tables 5 and 6 show the error in the natural frequencies and the parameter estimates after updating for different values of η and η_b. The value of α was computed using equation (16). The magnitude of the error bounds may be compared to $\|\mathbf{A}\|_2 = 20.68$ and $\|\mathbf{b}\|_2 = 1.27 \times 10^5$. Thus, the values of η_b of 10^4 and 2.5×10^4 represent 8 and 19% of the norm of **b**. The results are better for the smaller value of η_b for the same value of η, both in terms of the fit to the first nine natural frequencies and also in the prediction of the five higher natural frequencies. This may be expected as the errors in the measured data are likely to be nearer 8 than 19%.

TABLE 3

The natural frequencies for the GARTEUR structure

	Natural frequencies (Hz)						Errors (%)				
Mode	Test	Initial	$\eta = 0.1$	$\eta = 0.5$	$\eta = 1$	$\eta = 4$	Initial	$\eta = 0.1$	$\eta = 0.5$	$\eta = 1$	$\eta = 4$
f_1	6.38	6.61	6.47	6.46	6.42	6.24	3.53	1.47	1.23	0.70	− 2.26
f_2	16.10	16.95	16.30	16.28	16.31	16.22	5.25	1.24	1.14	1.33	0.72
f_3	33.10	35.98	33.62	33.69	33.51	33.77	8.70	1.56	1.80	1.24	2.02
f_4	33.53	36.45	33.97	34.08	33.77	33.92	8.72	1.31	1.63	0.71	1.17
f_5	35.65	37.22	35.05	35.07	35.13	35.81	4.40	− 1.69	− 1.64	− 1.45	0.44
f_6	48.38	50.88	48.52	48.49	48.49	47.66	5.17	0.28	0.24	0.23	− 1.49
f_7	49.43	53.94	49.15	49.03	48.77	52.18	9.12	− 0.57	− 0.81	− 1.33	5.56
f_8	55.08	59.30	55.20	55.20	55.26	57.01	7.66	0.21	0.22	0.32	3.51
f_9	63.04	64.48	63.77	63.67	63.63	63.72	2.29	1.15	1.00	0.93	1.08
f_{10}	66.50	66.94	64.35	64.35	64.35	64.61	0.66	− 3.23	− 3.24	− 3.24	− 2.85
f_{11}	102.90	105.71	98.94	98.94	100.44	104.40	2.73	− 3.85	− 3.85	− 2.39	1.46
f_{12}	130.54	133.09	127.53	127.51	128.35	130.49	1.95	− 2.30	− 2.32	− 1.68	− 0.04
f_{13}	141.38	146.33	140.97	140.63	139.93	137.17	3.50	− 0.29	− 0.53	− 1.03	− 2.98
f_{14}	151.32	158.68	149.17	148.91	148.76	150.41	4.86	− 1.42	− 1.59	− 1.69	− 0.60

TABLE 4

The updated parameters for the GARTEUR structure

Parameter	$\eta = 0.1$	$\eta = 0.5$	$\eta = 1$	$\eta = 4$
$(q_1)_t$	0.911	0.908	0.949	0.975
$(a_b)_w$	0.050	0.050	0.064	0.481
$(EI_y)_w$	1.098	1.091	1.073	0.934
$(GI_t)_w$	0.849	0.856	0.836	0.843
$(a_b)_t$	0.544	0.553	0.614	0.907
$(q_2)_{con}$	0.384	0.383	0.381	0.802

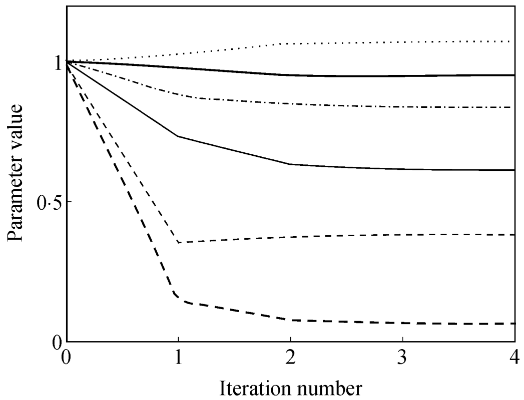

Figure 7. The evolution of the updated parameters with a constant η. —, $(q_1)_t$; – –, $(a_b)_w$; ......, EI_y; . – . –, GI_t; —, $(a_b)_t$; – –, $(q_2)_{con}$.

TABLE 5

The natural frequency errors for the GARTEUR structure using equation (16) to calculate α. The values of η_b should be multiplied by 1000

	Natural frequencies (Hz)		Errors (%)						
Mode	Test	Initial	Initial	$\eta = 0.5$ $\eta_b = 10$	$\eta = 1.0$ $\eta_b = 10$	$\eta = 4.0$ $\eta_b = 10$	$\eta = 0.5$ $\eta_b = 25$	$\eta = 1.0$ $\eta_b = 25$	$\eta = 4.0$ $\eta_b = 25$
f_1	6.38	6.61	3.53	1.43	− 1.54	− 2.25	− 1.67	− 1.80	− 2.28
f_2	16.10	16.95	5.25	1.02	1.01	0.73	0.98	0.94	0.71
f_3	33.10	35.98	8.70	2.02	0.98	2.02	1.07	1.18	2.16
f_4	33.53	36.45	8.72	2.14	0.14	1.17	0.22	0.33	1.31
f_5	35.65	37.22	4.40	− 1.65	− 0.33	0.43	− 0.21	− 0.08	0.48
f_6	48.38	50.88	5.17	0.25	− 1.19	− 1.48	− 1.26	− 1.33	− 1.47
f_7	49.43	53.94	9.12	− 0.60	1.61	5.54	2.10	2.73	5.78
f_8	55.08	59.30	7.66	0.22	1.56	3.51	1.83	2.12	3.69
f_9	63.04	64.48	2.29	0.98	0.58	1.08	0.64	0.71	1.12
f_{10}	66.50	66.94	0.66	− 3.24	− 3.12	− 2.85	− 3.10	− 3.06	− 2.82
f_{11}	102.90	105.71	2.73	− 5.02	0.01	1.45	0.20	0.42	1.56
f_{12}	130.54	133.09	1.95	− 2.76	− 0.65	− 0.04	− 0.57	− 0.48	0.00
f_{13}	141.38	146.33	3.50	− 0.33	− 2.86	− 2.98	− 2.90	− 2.95	− 2.95
f_{14}	151.32	158.68	4.86	− 1.55	− 1.70	− 0.60	− 1.56	− 1.40	− 0.49

TABLE 6

The updated parameters for the GARTEUR structure using equation (16) to calculate α. The values of η_b should be multiplied by 1000

Parameter	$\eta = 0.5$ $\eta_b = 10$	$\eta = 1.0$ $\eta_b = 10$	$\eta = 4.0$ $\eta_b = 10$	$\eta = 0.5$ $\eta_b = 25$	$\eta = 1.0$ $\eta_b = 25$	$\eta = 4.0$ $\eta_b = 25$
$(q_1)_t$	0.868	0.966	0.975	0.966	0.966	0.977
$(a_b)_w$	0.050	0.230	0.481	0.265	0.302	0.503
$(EI_y)_w$	1.097	0.986	0.935	0.978	0.969	0.931
$(GI_t)_w$	0.867	0.823	0.843	0.825	0.827	0.845
$(a_b)_t$	0.527	0.812	0.907	0.829	0.846	0.913
$(q_2)_{\text{con}}$	0.383	0.528	0.799	0.551	0.587	0.818

8. CONCLUSIONS

This paper has introduced the robust model updating approach, where the uncertainty in the model and measurements are specified as bounds on the norms of the coefficient matrix and the residual vector. This is shown to be equivalent to Tikhonov regularisation and the relationship between the uncertainty bound and the Tikhonov regularisation parameter has been explored. Two approaches to obtaining the uncertainty bound were given, namely using the corner of the 'L' curve, and using the expected parameter uncertainty. The robust updating approach was demonstrated on a simulated cantilever beam example in order to highlight the statistical properties of the approach, and also on the experimental data from the GARTEUR test structure.

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