Clifford Algebraic Perspective on Second-Order Linear Systems

Seamus D. Garvey*
Aston University, Birmingham, England B4 7ET, United Kingdom

Michael I. Friswell†
University of Wales Swansea, Swansea, Wales SA2 8PP, United Kingdom

and

John E. T. Penny‡
Aston University, Birmingham, England B4 7ET, United Kingdom

A substantial proportion of all dynamic models arising naturally present themselves initially in the form of a system of second-order ordinary differential equations. Despite this, the established wisdom is that a system of first-order equations should be used as a standard form in which to cast the equations characterizing every dynamic system and that the set of complex numbers, and its algebra, should be used in dynamic calculations, particularly in the frequency domain. For any dynamic model occurring naturally in second-order form, it is proposed that it is both intuitively and computationally sensible not to transform the model into state-space form. Instead, it is proposed that Clifford algebra, $Cl_{2}$, be used in the representation and manipulation of this system. The attractions of this algebra are indicated in three contexts: 1) the concept of similarity transformations for second-order systems, 2) the solution for characteristic roots of self-adjoint systems, and 3) a model reduction for finite element models.

Nomenclature

- $A, B, C$ = state-space matrices representing a first-order system in Eq. (49)
- $\tilde{A}, \tilde{B}, \tilde{C}$ = second order state-space matrices for a second-order system
- $D'$ = damping matrix in nearest classically damped system
- $E_{L}, F_{L}, G_{L}, H_{L}$ = partitions of the left $(2N \times 2N)$ modal matrix in Eq. (5)
- $E_{R}, F_{R}, G_{R}, H_{R}$ = partitions of the right $(2N \times 2N)$ modal matrix in Eq. (5)
- $J$ = $(2N \times 2N)$ matrix for removing all complexity from Eq. (5)
- $K, D, M$ = system stiffness, damping, and mass matrices
- $K_{qq}, D_{qq}, M_{qq}$ = system matrices appropriate to coordinate system $[q, Q]$;
- $K_{rr}, D_{rr}, M_{rr}$ = system matrices appropriate to coordinate system $[r, R]$
- $L$ = arbitrary integer power used in Eqs. (39), (51), and (52)
- $N$ = number of degrees of freedom in the original system
- $P$ = number of pairs of complex roots in the original system
- $P, N$ = $(2N \times 2N)$ matrices characterising the second order system
- $P_{qq}, N_{qq}$ = $(2N \times 2N)$ matrices $P, N$ for coordinate system $[q, Q]$
- $P, N$ = Clifford system matrices [corresponding to $P$ and $N$ of Eq. (19)]
- $Q$ = number of pairs of real roots in the original system, $N - P$
- $Q, q$ = vectors of generalized displacements and forces
- $R, r$ = vectors of generalized displacements and forces

$S$ = matrix to select degrees of freedom of interest

- $S_{1}, S_{2}$ = two diagonal $(N \times N)$ matrices of characteristic roots [from Eq. (5)]
- $\tilde{T}_{L}, \tilde{T}_{R}$ = Clifford coordinate transformation matrices
- $T_{L}, T_{R}$ = left and right coordinate transformation matrices
- $T, t$ = vectors of displacements and forces at terminal degrees of freedom
- $\tilde{U}$ = Clifford modal matrix containing modal data for the system
- $W_{L}, X_{L}, Y_{L}, Z_{L}$ = partitions of the left $(2N \times 2N)$ modal matrix in Eq. (17)
- $W_{R}, X_{R}, Y_{R}, Z_{R}$ = partitions of the right $(2N \times 2N)$ modal matrix in Eq. (17)
- $\Gamma$ = $(2N \times 2N)$ matrix comprising four diagonal $(N \times N)$ partitions
- $\gamma$ = $(N \times N)$ diagonal matrix from which $\Gamma$ is derived according to Eq. (12)
- $\Theta(\omega)$ = $(N \times N)$ receptance matrix
- $\Lambda$ = $(N \times N)$ diagonal matrix containing the eigenvalues of $A$ [Eq. (40)]
- $\Phi_{L}, \Phi_{R}$ = left and right modal matrices of the undamped system, $(K, M)$
- $\Omega$ = diagonal matrix of undamped natural frequencies, rad/s
- $\tilde{\Omega}$ = Clifford spectral matrix [corresponding to right-hand side of Eq. (17a)]
- $\tilde{I}, \tilde{i}, \tilde{j}, \tilde{k}$ = elementary components of Clifford algebra, $Cl_{2}$ [see Eq. (27)]
- $2\zeta\Omega$ = diagonal matrix of damping terms, rad/s

I. Introduction

The discretized model of a general second-order system takes the form

$$Kq + Dq + M\ddot{q} = \dot{Q}$$

(1)

Vector $q$ represents model displacements as a function of time and $\dot{Q}$ represents corresponding forces. If each contains $N$ entries, we refer to Eq. (1) as an $N$-degree-of-freedom second-order model. There are numerous possible functions for such a model. We might require to do the following:

1) Simulate the model in the time domain given known $Q(t)$.
2) Determine steady-state response of the model to harmonic forcing over a range of frequencies.
3) Determine the characteristic roots and associated mode shapes from the system matrices.

4) Invert the model to determine \( Q(t) \) given only information about \( q(t) \) or given a mix of information about \( q(t) \) and \( Q(t) \).

5) Compute either the complete system matrices or update parameters controlling these matrices using knowledge of both the input and output functions.

6) Devise controllers with \( Q(t) \) comprising a component due to feedback based on \( q(t) \) (structural modifications are effectively like feedback controllers).

7) Examine the sensitivity of response in the time or frequency domain to changes in the parameters that govern the system matrices.

In most of the preceding applications, it is either strongly advantageous or mandatory to deal with the eigenpairs of the system (the characteristic roots and associated characteristic vectors). In the case of an undamped system \((D = 0)\), the eigenpairs are found by solving the generalized eigenvalue problem:

\[
\Phi_L^T K \Phi_R = \Omega^2 (\text{diagonal}), \quad \Phi_L^T M \Phi_R = I \tag{2}
\]

When the undamped problem is self-adjoint, the system matrices are symmetrical and the matrices of left and right modes, \( \Phi_L \) and \( \Phi_R \) respectively, are identical. Provided that \( M \) is positive definite, for all undamped systems, we can find real \( \Omega \) and \( \Phi_R \) to satisfy Eq. (2).

In a very small proportion of damped systems, it transpires that the same modal matrices that diagonalize both the mass and stiffness matrices also diagonalize the damping matrix. That is,

\[
\Phi_L^T D \Phi_R = 2 \Omega \Omega (\text{diagonal}) \tag{3}
\]

The notations \( \Omega^2 \) and \( 2 \Omega \) are chosen deliberately to invoke the obvious connections with the scalar quantities \( \omega_0 \) and \( 2 \omega_0 \), used ubiquitously in single-degree-of-freedom second-order systems. Second order systems that can simultaneously satisfy Eqs. (2) and (3) are said to have classical or proportional damping. Some researchers argue that the term proportional damping implies that the damping matrix \( D \) is formed as a scalar combination of the stiffness and mass matrices, and for this reason, we prefer to use the term classical damping \(^1\) in connection with any second-order system that can simultaneously satisfy Eqs. (2) and (3).

In practice, it is extraordinary for systems to be truly classically damped. There is a substantial body of literature \(^2\) relating to either quantifying (or bounding) the errors that occur through assuming that a given general second-order system is classically damped or to obtaining the true eigenpairs of the generally damped system beginning with the eigenpairs of \( \{K, D, M\} \), where \( D \) is chosen such that \((\Phi_L^T D \Phi_R)\) is diagonal and its diagonal entries are equal to those of \((\Phi_L^T K \Phi_R)\).

The process of determining the eigenpairs of any sizable system is not direct in the sense that it involves the solution of an eigenvalue problem and this always involves iteration for \( N > 5 \). If we require to solve for the eigenpairs of a generally damped second-order system using methods that are direct, an equivalent 2N-degree-of-freedom first-order system is formed whose characteristic roots are identical to the characteristic roots of the second-order system. Note that although the solution might be direct, iteration will be involved in the solution of the eigenvalue problem. There are many possible choices in the formation of such first-order systems. The following two general forms illustrate this. In these, we use the notation \( s \) to represent at once the Laplace frequency variable and the differentiation operator with respect to time,

\[
\begin{align*}
(0 \ Y M \ Y D)(p) & + (X \ 0 \ Y M \ Y D)(p) = 0 \\
(0 \ Y K \ Y D)(q) & - (X \ 0 \ Y K \ Y D)(q) = 0 \\
\end{align*}
\tag{4a}
\]

In Eqs. (4a) or (4b), \( X \) and \( Y \) can be set to any arbitrary square nonsingular matrices. One could achieve further generality by splitting the contribution of the \( D \) matrix in any arbitrary way between the two \((2N \times 2N)\) matrices and by premultiplying either Eq. (4a) or Eq. (4b) by an arbitrary nonsingular \((2N \times 2N)\) matrix. In this paper, we shall favor the use of Eq. (4b) with \( Y = I \) and \( X = K \). This arrangement permits us to retain symmetry in the two matrices when the system itself is symmetric. It has the slight disadvantage that \( K \) is not always nonsingular; freely floating objects will produce singular \( K \) matrices. However, it is usually possible to decouple the rigid-body motion and, hence, to consider a reduced model with a nonsingular \( K \).

II. Recasting the Equation for the Characteristic Roots

By the use of our preferred first-order form, the eigenvalue problem providing solutions for the characteristic roots of Eq. (1) may be written as

\[
\begin{align*}
\begin{bmatrix}
E_L & F_L \\
G_L & H_L
\end{bmatrix}^T
\begin{bmatrix}
0 & K \\
D & 0
\end{bmatrix}
\begin{bmatrix}
E_R & F_R \\
G_R & H_R
\end{bmatrix} &= \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \tag{5a}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
E_L & F_L \\
G_L & H_L
\end{bmatrix}^T
\begin{bmatrix}
K & 0 \\
0 & -M
\end{bmatrix}
\begin{bmatrix}
E_R & F_R \\
G_R & H_R
\end{bmatrix} &= \begin{bmatrix} I_{(N \times N)} & 0 \\ 0 & I_{(N \times N)} \end{bmatrix} \tag{5b}
\end{align*}
\]

Equations (5a) and (5b) find the left characteristic vectors as well as the right. Matrices \( S_1 \) and \( S_2 \) are diagonal. Where complex roots occur, they occur in conjugate pairs. If \( 2P \) of the \( 2N \) roots are complex, then we may arrange the roots and vectors such that \( S_2(k, k) = \text{conj}[S_1(k, k)] \) for \( k \leq P \). Where real roots occur, they do so in two distinct groups, and we arrange them accordingly. For one-half of the real roots, the associated right vectors [arranged in columns \((P + 1:N)\) of \( E_R \) and \( G_R \)] comprise purely real entries. For the other half of the real roots, the associated right vectors [arranged in columns \((P + 1:N)\) of \( F_R \) and \( H_R \)] comprise purely imaginary entries. Similar statements apply to the left vectors except that these appear in \( E_L, F_L, G_L \), and \( H_L \).

Define \( Q = N - P \). The total number of real roots found is \( 2Q \). Now define a useful matrix \( J \) as

\[
J = \begin{bmatrix}
1 & \sqrt{2} & 0 & 0 & I_{(N \times Q)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
J^T J = \begin{bmatrix} I_{(N \times N)} & 0 \\ 0 & -I_{(N \times N)} \end{bmatrix} \tag{6a}
\]

Postmultiply Eq. (5) by \( J \) and premultiply it by \( J^T \). All imaginary components are eliminated from the equations by this action, and we obtain real matrices \( W, X, Y, \) and \( Z \) (in right and left versions) and \( S_1, S_2, S_3, S_4, S_5, S_6, S_7, \) and \( S_8 \):

\[
\begin{align*}
\begin{bmatrix}
W_L & \begin{bmatrix} X_L \\
Y_L \end{bmatrix}^T \\
\end{bmatrix}^T
\begin{bmatrix}
0 & K \\
D & 0
\end{bmatrix}
\begin{bmatrix}
W_R & \begin{bmatrix} X_R \\
Y_R \end{bmatrix}^T \\
\end{bmatrix} &= \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \\
\end{bmatrix} \tag{7a}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
W_L & \begin{bmatrix} X_L \\
Y_L \end{bmatrix}^T \\
\end{bmatrix}^T
\begin{bmatrix}
K & 0 \\
0 & -M
\end{bmatrix}
\begin{bmatrix}
W_R & \begin{bmatrix} X_R \\
Y_R \end{bmatrix}^T \\
\end{bmatrix} &= \begin{bmatrix} I_{(N \times N)} & 0 \\ 0 & -I_{(N \times N)} \end{bmatrix} \tag{7b}
\end{align*}
\]

The primes are used here because we shall define matrices \( W, X, Y, \) and \( Z \) (left and right versions) slightly differently due in course. We pause to remark on the contents of \( S_1, S_2, S_3, \) and \( S_4, \) all of which are real and diagonal:

\[
\begin{align*}
S_1(k, k) &= \text{Re}(S_1(k, k)) \leq P \\
S_2(k, k) &= S_1(k, k) > P \\
S_3(k, k) &= \text{Im}(S_1(k, k)) \leq P \\
S_4(k, k) &= 0.0 > P \leq P \\
S_5(k, k) &= -S_1(k, k) \leq P \\
S_6(k, k) &= -S_1(k, k) > P \\
\end{align*}
\tag{8}
\]
Recall that $S_3(k, k) = \text{conj}[S_3(k, k)]$ for all $k \leq P$. The following identity is helpful in verifying Eqs. (7) and (8):

$$
\begin{bmatrix}
1 / \sqrt{2} & 1 / \sqrt{2} \\
-j / \sqrt{2} & j / \sqrt{2}
\end{bmatrix}
\begin{bmatrix}
(a + jb) & 0 \\
0 & (a - jb)
\end{bmatrix}
\begin{bmatrix}
1 / \sqrt{2} & -j / \sqrt{2} \\
j / \sqrt{2} & j / \sqrt{2}
\end{bmatrix}
= \begin{bmatrix} a & b \\ b & -a \end{bmatrix}
$$

(9)

Suppose Eq. (7b) is modified such that $I_{(2N \times 2N)}$ appears on the right side and the $\sim$ sign is removed from $-M$. By obvious manipulations

$$
\begin{align}
W'_L X'_L Z'_L & = \begin{bmatrix} K & 0 \\ D & M \end{bmatrix} \begin{bmatrix} W_R X'_R Z'_R = S_3 & S_1 \\ S_1 & S_3 \end{bmatrix} \\
W'_L X'_L Z'_L & = \begin{bmatrix} I_{(N \times N)} & 0 \\ 0 & I_{(N \times N)} \end{bmatrix}
\end{align}
$$

(10a)

Equation (10) motivates the introduction of a notation for the operation of negating the off-diagonal blocks of a block ($2 \times 2$) matrix. For reasons which will become evident later, we denote this operator $\text{conj}_{\text{jk}}()$. It is straightforward to show that, for any arbitrary block ($2 \times 2$) matrices $Y$ and $Z$,

$$
\text{conj}_{\text{jk}}(YZ) = \text{conj}_{\text{jk}}(Y)\text{conj}_{\text{jk}}(Z)
$$

(11)

Define a new ($2N \times 2N$) matrix $\Gamma$ in terms of diagonal ($N \times N$) matrix $\gamma$ as

$$
\Gamma = \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix}
$$

(12)

From the earlier definition of the $\text{conj}_{\text{jk}}()$ operator, we can easily verify that for any $\gamma$

$$
\Gamma^T \text{conj}_{\text{jk}}(\Gamma) = I_{(2N \times 2N)}
$$

(13)

We can always select $\Gamma$ such that

$$
\Gamma^T \begin{bmatrix} S_3 & S_1 \\ S_1 & S_3 \end{bmatrix} \Gamma = \begin{bmatrix} 0 & \Omega \\ \Omega & 2\zeta\Omega \end{bmatrix}
$$

(14)

Because $\Gamma$ comprises four ($N \times N$) diagonal blocks and $S_3$, $S_1$, and $S_1$ are all diagonal, matrices $\Omega$ and $\zeta$ must also be diagonal. The use of the notation $\Omega$ and $\zeta$ is consistent with its use earlier. If the system was classically damped, we could obtain the same matrices by solving the eigenvalue problem for the undamped system first and then performing the modal transformation on the system damping matrix. The determination of individual entries in $\gamma$ is simple using

$$
\begin{align}
\cosh(\gamma) & \sinh(\gamma) \\
\sinh(\gamma) & \cosh(\gamma)
\end{align}
\begin{bmatrix} a + c & b \\ b & c - a \end{bmatrix}
= \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + b \begin{bmatrix} \cosh(2\gamma) & \sinh(2\gamma) \\ \sinh(2\gamma) & \cosh(2\gamma) \end{bmatrix}
$$

(15)

We now define (right and left versions of) $W, X, Y, Z$ based on $W', X', Y', Z'$ and $\Gamma$; thus,

$$
\begin{bmatrix} W_L & X_L \\ Y_L & Z_L \end{bmatrix} = \begin{bmatrix} W'_L & X'_L \\ Y'_L & Z'_L \end{bmatrix} \Gamma
$$

(16)

Applying Eqs. (13), (14), and (16) leads us to the following key result:

$$
\begin{align}
(W_L & W_L \ X_L & K \ D) \begin{bmatrix} W_R & X_R \\ Y_R & Z_R \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega & 2\zeta\Omega \end{bmatrix} \\
W_L & W_L \ X_L & (K \ 0) \begin{bmatrix} W_R & X_R \\ Y_R & Z_R \end{bmatrix} = \begin{bmatrix} I_{(N \times N)} & 0 \\ 0 & I_{(N \times N)} \end{bmatrix}
\end{align}
$$

(17a)

(17b)

Equations (17a) and (17b) tells us that we can find a real generalized version of a coordinate transformation that decouples a representation of our original $N$-degree-of-freedom second-order system into $N$-single-degree-of-freedom second-order systems. We shall refer to Eq. (17) as the Clifford formulation of the equation for characteristic roots.

### III. Second-Order Coordinate Transformations

In analyzing undamped vibrations, extensive use is made of coordinate transformations. Given an original vector of displacement coordinates $q$, associated force vector $Q$, and system mass and stiffness matrices $M_q$ and $K_q$, respectively, we can use coordinate-transformation matrices $(T_q, T_R)$ as follows:

$$
q = T_q r, \quad R = T_R^T Q
$$

(18)

If $(T_q, T_R)$ are square and invertible, then the natural frequencies computed from $[M_q, K_q]$ will be identical to those computed from $[M_q, K_q]$. Moreover, the mode shapes computed from $[M_q, K_q]$ can all be premultiplied by $T_R$ to give precisely the same mode shapes as would be computed from $[M_q, K_q]$. For these reasons, we are justified in referring to any such transformation comprising square and invertible $(T_q, T_R)$ as a generalised similarity transformation. When the original system matrices are all symmetrical, we invariably use $T_q = T_R$. Many practitioners adhere to this constraint even when the system matrices are not symmetrical.

Not all coordinate transformations are similarity transformations. Frequently, coordinate transformations are used to impose constraints. In these cases, $(T_q, T_R)$ have fewer columns than rows. All model-reduction methods effectively comprise the imposition of constraints. The Guyan (or static) reduction, the system equivalent reduction expansion process (SEREP) reduction, the iterated improved reduced system (IIRS) reduction, the representation of substructures using fixed-constraint modes and attachment modes, and the modal-reduction are all examples of model-reducing transformations that are independent of frequency. There is a body of literature on the expansion of operating shapes and mode shapes that is effectively a perfect reverse of model reduction with the slight qualification that the transformation matrices in these applications are often frequency dependent.

Guyan reduction preserves static flexibility exactly. SEREP reduction preserves perfect reproducibility of a subset of the natural frequencies and associated mode shapes. The Craig–Bampton representation preserves static flexibility exactly for a subset (the attachment subset) of the model (substructure) degrees of freedom and preserves exactly the internal modes of the substructure. All of these methods can be extended into second-order form using Eq. (17).

Accept that a general $N$-degree-of-freedom second-order system can be fully represented by the $(2N \times 2N)$ matrices $[P_{qq}, N_{qq}]$ and that some generalized coordinate transformation matrices $(T_q, T_R)$ exist having dimensions $(2N \times 2N)$:

$$
\begin{align}
P_{qq} &= \begin{bmatrix} 0 & K_{qq} \\ K_{qq} & D_{qq} \end{bmatrix}, & N_{qq} &= \begin{bmatrix} 0 & M_{qq} \\ M_{qq} & 0 \end{bmatrix} \\
T_q &= \begin{bmatrix} T_{q11} & T_{q12} \\ T_{q21} & T_{q22} \end{bmatrix}, & T_R &= \begin{bmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{bmatrix}
\end{align}
$$

(19)
It follows from Eqs. (11) and (17) that if we allow \( M = N \) and if we insist that \( \{ T_L, T_R \} \) are both nonsingular, a second-order system \( \{ P_{rr}, N_{rr} \} \) having identical characteristic roots and related characteristic vectors can be formed as

\[
P_{rr} = T_L^T P_{eq} T_R, \quad N_{rr} = \text{conj.} j (T_L^T N_{eq} T_R)
\]

(20)

Of course, if we choose completely arbitrary matrices \( T_L, T_R \), a great deal of structure in the form of \( P, N \) is lost. Specifically, 1) the off-diagonal blocks of \( N \) may no longer be zeros, 2) block (1, 1) in \( N \) may not be equal to block (1, 2) of \( P \), and 3) block (1, 1) in \( P \) may not be zero.

In a later section, we address these important matters and we examine, in particular, whether we can begin from fully general \((2N \times 2N)\) matrices \( P \) and \( N \) and find transformations that will restore these three features.

For the moment, however, we shall not dwell on the possible loss of structure. Equation (20) provides a general expression for second-order coordinate transformations. The new coordinate set to which we transform is implicit from

\[
\begin{pmatrix} q \\
p \end{pmatrix} = \begin{pmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{pmatrix} \begin{pmatrix} t \\
r \end{pmatrix}
\]

(21)

The original coordinate vector contained the vector \( q \) and its derivative with respect to time, \( p \). The new coordinate vector does not have this simple relationship between its two parts. This need not concern us. Note also that in Eq. (21) we have written the relationship between the original displacement-type, that is, displacement and velocity, coordinate vector and the new. We have not written the relationship between the original force-type coordinate vector and the new. This is a deliberate omission rectified in a later section.

We conclude this section by observing that we can find the general second-order equivalents of Guyan reduction\(^9\) and SEREP. Guyan’s original paper focused on systems having symmetrical \( K \), which implicitly uses \( T_L = T_R \). In the general case of unsymmetrical \( K \), we shall allow \( T_L \) and \( T_R \) to be defined separately. Guyan’s paper also implicitly asserts that no external forces can be applied to the slave (discarded) degrees of freedom. This assertion is also unnecessary, and we generalize here. The vector of displacements \( q \) is partitioned into master (retained) and slave degrees of freedom. The stiffness matrix is partitioned similarly,

\[
K = \begin{pmatrix} K_{mm} & K_{mr} \\ K_{rm} & K_{rr} \end{pmatrix}, \quad q = \begin{pmatrix} q_m \\ q_r \end{pmatrix}, \quad Q = \begin{pmatrix} Q_m \\ Q_r \end{pmatrix}
\]

\[
q_m = \begin{pmatrix} I \\ -K_{rr}^{-1} K_{rm} \end{pmatrix} r \equiv T_R r
\]

\[
R^T = Q^T \begin{pmatrix} I \\ -K_{rr}^{-1} \end{pmatrix} T_L
\]

(22)

Reduced stiffness, damping, and mass matrices can be computed as

\[
K_r = (T_L^T K T_R), \quad D_r = (T_L^T D T_R), \quad M_r = (T_L^T M T_R)
\]

(23)

Note that using Eqs. (22) and (23) we can begin with a force vector that includes some forces on the slave degrees of freedom and compute an estimate of the response vector including the retained degrees of freedom. Hence, at any given frequency \( \omega \), we can produce an approximation \( \Theta(\omega) \) to the full-size receptance matrix \( \Theta(\omega) \). Accepting that \( \Theta(\omega) \) can be partitioned similarly to \( K \) in (22), we find that at zero frequency (\( \omega = 0 \)) \( \Theta_{mm}(\omega), \Theta_{mr}(\omega), \) \( \Theta_{rm}(\omega) \) and \( \Theta_{rr}(\omega) \) are all approximately equal. At low frequencies (compared with the natural frequencies of the slave system \( \{ K_r, M_r \} \)), the errors, \( [\Theta_{mm}(\omega) - \Theta_{mm}(0)] \) are found to be proportional to \( \omega^2 \), whereas both \( [\Theta_{mr}(\omega) - \Theta_{mr}(0)] \) and \( [\Theta_{rm}(\omega) - \Theta_{rm}(0)] \) are proportional to \( \omega \).

The second-order equivalent of Guyan reduction\(^9\) involves both the stiffness and damping matrices (as contained in \( P_{eq} \)). Using the same partitioning as Eq. (22) leads to a more complex partitioning of \( P_{eq} \). Ignoring that \( P_{eq} \) has some particular structure,

\[
P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad P_{eq} = \begin{pmatrix} P_{11,eq} & P_{12,eq} \\ P_{21,eq} & P_{22,eq} \end{pmatrix}
\]

\[
t_R = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad t_{eq} = \begin{pmatrix} P_{11,eq} & P_{12,eq} \\ P_{21,eq} & P_{22,eq} \end{pmatrix}
\]

\[
T_R = \begin{pmatrix} 1 & 0 \\ t_{R11} & t_{R12} \\ 0 & I \end{pmatrix}, \quad T_{eq} = \begin{pmatrix} 1 & 0 \\ t_{eq11} & t_{eq12} \\ 0 & I \end{pmatrix}
\]

(24)

If the transformation matrix \( T \) derived from Eq. (24) is applied to \( P_{eq} \) and \( N_{eq} \) as defined in Eq. (19) such that new matrices \( \{ P_{rr}, N_{rr} \} \) are produced, we shall have carried out the second-order equivalent of Guyan reduction.

It transpires that this extension of Guyan reduction\(^9\) preserves the original structure of the matrices \( P \) and \( N \) and that no restoration of form is necessary. If implemented in a computer code, there are no negative implications on memory requirements because the system is always fully represented by three matrices, \( K, D, \) and \( M \). We defer the proof of this point to another paper. Moreover, we demonstrate in an example that the receptance errors, \( [\Theta_{mm}(\omega) - \Theta_{mm}(0)], [\Theta_{mr}(\omega) - \Theta_{mr}(0)], \) and \( [\Theta_{rm}(\omega) - \Theta_{rm}(0)] \) are all proportional to \( \omega^2 \), even for generally damped systems.

We now turn briefly to the extension of SEREP. For undamped systems, SEREP involves solving the generalized eigenvalue problem given by \( \{ K_r, M_r \} \) and selecting a subset of the computed mode shapes to be retained. The retained subset of the right mode shapes are given as \( \Phi_r \), which may be split into the master and slave degrees of freedom as

\[
\Phi_R = \begin{pmatrix} \Phi_{Rm} \\ \Phi_{Rr} \end{pmatrix}
\]

Similarly, the retained left mode shapes may be written as

\[
\Phi_L = \begin{pmatrix} \Phi_{Lm} \\ \Phi_{Lr} \end{pmatrix}
\]

Assuming that there are more master degrees of freedom than retained mode shapes, we have the left and right reduction transformations as follows:

\[
T_R = \begin{pmatrix} \Phi_{Rm} \\ \Phi_{Rr} \end{pmatrix}, \quad \Phi_{Rm}^{-1} \Phi_{Rr} = \begin{pmatrix} \Phi_{Em} \\ \Phi_{Er} \end{pmatrix}
\]

(25)

When nonclassical damping is present, we can use the transformation in Eq. (17) to extend the SEREP method to second-order form. The \( W, X, Y, \) and \( Z \) matrices are split into master and slave degrees of freedom, but also only those columns are retained that correspond to the eigenvalues and eigenvectors of interest. Thus, a decision is made to retain elements of \( \Omega \) and the corresponding elements of \( \zeta \), which then determines the columns of \( W, X, Y, \) and \( Z \) to retain. The transformations are then given by

\[
\begin{pmatrix} W_{Rm} & X_{Rm} \\ W_{Rr} & X_{Rr} \end{pmatrix}, \begin{pmatrix} W_{Rm} & X_{Rm} \\ Y_{Rm} & Z_{Rm} \end{pmatrix}
\]

(26)
and similarly for \( T_L \). The superscripted plus in Eq. (26) is the same Moore–Penrose pseudoinverse as is defined implicitly in Eq. (25). As before, \( m \) and \( s \) are master and slave degrees of freedom, and \( r \) are the retained modes. The extension of SEREP to the second-order case results in a loss of structure from \( P \) and \( N \). We have committed to examining the possibilities of restoring this structure, and it is obvious that if/when this is possible, we can use established methods for dealing with the (restored-structure) reduced models. Although there are several strong motivations for restoring the structure, it is important to realize that this restoration is not necessary. Using the concepts presented in the following section, we shall see that we can efficiently solve characteristic value problems such as Eq. (17) in cases when the \( P \) and \( N \) matrices do not have this structure.

### IV. Role of Clifford Algebra, \( \text{Cl}_2 \)

Clifford algebra dates back to 1872 when William Kingdon Clifford presented a structure for a family of algebras to the London Mathematical Society. Each of these algebras is founded on a vector space of finite dimension. In the general case where a vector space of dimension \( n \) is used, the algebra is called \( \text{Cl}_n \), and it is comprised of numbers having \( 2^n \) parts. The algebras have a very attractive geometric quality, and they are distinguished from previous attempts at algebras for vector spaces by the existence of an inverse. Our interest is specifically in \( \text{Cl}_2 \), and our present use of \( \text{Cl}_2 \) arises only because of its algebraic behavior, although ultimately we hope to attach significance to its geometrical elements. We cannot devote space to explaining the mechanics of Clifford algebra in general, but we strongly commend to any engineering reader some study of the generality of Clifford algebra. An appreciation of the essential mechanics of the algebra is easily possible within a few hours through appropriate texts.\(^{13-15}\) The importance of Clifford algebra is continually being uncovered, and there is substantial evidence to suggest that knowledge of Clifford algebra may one day supplant a host of other mathematical devices.

Each number in the Clifford algebra, \( \text{Cl}_2 \), has four components just as each complex number has two. The basis elements are ordinarily denoted \([1, e_1, e_2, e_1 e_2]\). Because we wish to use superscripts to pair coefficients implicitly with the basis components, we deliberately use the notation \( 1, i, j, \) and \( k \) instead. For the purposes of this paper, each of these quantities represents one of four fundamental components of any general \((2 \times 2)\) matrix:

\[
\begin{align*}
\hat{1} &:: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
i &:: \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
j &:: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
k &:: \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\] (27)

In Eq. (27) and in later equations, the notation \( :: \) is used to imply a correspondence. Any \((2 \times 2)\) matrix can be represented as a linear combination of these four components, and we can say that the Clifford algebra, \( \text{Cl}_2 \), is isomorphic to the algebra of \((2 \times 2)\) matrices over the field of real numbers.\(^{16}\) In this paper, we are not interested in any Clifford algebra other than \( \text{Cl}_2 \) and whenever the term Clifford number arises, it will invariably mean a Clifford number from the algebra \( \text{Cl}_2 \).

The general Clifford number \( \hat{x} \) is written as

\[
\hat{x} = x_0 + x_1 \hat{1} + x_j \hat{i} + x_k \hat{j}
\] (28)

The rules of multiplication for \( \text{Cl}_2 \) are readily deduced from the behavior of the fundamental components of the general \((2 \times 2)\) matrix described in Eq. (27). These are summarized in Table 1.

<table>
<thead>
<tr>
<th>First operand</th>
<th>Second operand</th>
<th>Multiplication table for the components of ( \text{Cl}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{i} )</td>
<td>( \hat{j} )</td>
<td>( \hat{i} \hat{j} )</td>
</tr>
<tr>
<td>( \hat{j} )</td>
<td>( \hat{k} )</td>
<td>( \hat{k} \hat{j} )</td>
</tr>
</tbody>
</table>

Because the rules of multiplication for Clifford numbers are based on matrix multiplication, we will automatically find that, for any matrices \( P \) and \( Q \), the Clifford representation \( PQ \) of the product \( PQ \) is equal to \( P \) multiplied by \( Q \). The concept of a conjugate is important with the Clifford numbers in this context. It is possible to define several different conjugates for the Clifford numbers. Let \( \hat{X} = X_0 \hat{1} + X_j \hat{i} + X_k \hat{j} + X_k \hat{k} \) then

\[
\begin{align*}
\text{conj}_1(\hat{X}) &= X_0 \hat{1} - X_j \hat{i} + X_k \hat{j} - X_k \hat{k} \\
\text{conj}_2(\hat{X}) &= X_0 \hat{1} + X_j \hat{i} + X_k \hat{j} - X_k \hat{k} \\
\text{conj}_{i,k}(\hat{X}) &= X_0 \hat{1} - X_i \hat{i} + X_j \hat{j} - X_k \hat{k}
\end{align*}
\] (30)

Just as conjugation tends to be associated with transposition when dealing with matrices of complex numbers, it is convenient to define the two different conjugate-transpose operators, \((\cdot)^*\) and \((\cdot)^\dagger\), by

\[
\begin{align*}
\hat{X}^* &= X_0^* \hat{1} - X_j^* \hat{i} + X_k^* \hat{j} - X_k^* \hat{k} \\
\hat{X}^\dagger &= X_0^* \hat{1} + X_j^* \hat{i} + X_k^* \hat{j} - X_k^* \hat{k}
\end{align*}
\] (31)

It is instructive to see the effects of the conjugates and the conjugate transposes on the \((2N \times 2M)\) matrices that they represent. Allowing a mixture of notation for the sake of brevity, we have

\[
\begin{align*}
\text{conj}_1\left( \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \right) &= \begin{pmatrix} W & -Y \\ -X & Z \end{pmatrix} \\
\text{conj}_2\left( \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \right) &= \begin{pmatrix} W & Y \\ X & Z \end{pmatrix} \\
\text{conj}_{i,k}\left( \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \right) &= \begin{pmatrix} W & -X \\ -Y & Z \end{pmatrix} (\hat{W}^* - \hat{Y}^*) \\
\left( \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \right)^* &= \begin{pmatrix} W^T & -Y^T \\ -X^T & Z^T \end{pmatrix} \\
\left( \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \right)^\dagger &= \begin{pmatrix} W^T & Y^T \\ X^T & Z^T \end{pmatrix}
\end{align*}
\] (32)

Recognizing the effects shown in Eqs. (32), we can write Eq. (17) as

\[
\begin{align*}
\hat{U}_L &= \begin{pmatrix} W_L & X_L \\ Y_L & Z_L \end{pmatrix}, \\
\hat{U}_R &= \begin{pmatrix} W_R & X_R \\ Y_R & Z_R \end{pmatrix}, \\
\hat{P} &= \begin{pmatrix} 0 & K \\ K & D \end{pmatrix}, \\
\hat{N} &= \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \\
\hat{I} &= \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}
\end{align*}
\] (33)

where the following correspondences apply:

\[
\begin{align*}
\hat{U}_L &= \hat{U}_L, \\
\hat{U}_R &= \hat{U}_R, \\
\hat{P} &= \hat{P}, \\
\hat{N} &= \hat{N}, \\
\hat{I} &= \hat{I}
\end{align*}
\] (34)

Solution of Eq. (17) can be achieved through a series of similarity transformations (second-order coordinate transformations as described in the preceding section). Work is continuing in the solution methods. This leads to the definition of \( \hat{U}_L \) and \( \hat{U}_R \) as

\[
\hat{U}_R = \prod_{i=0}^{\infty} T_{R,i}^{\dagger}, \\
\hat{U}_L = \prod_{i=0}^{\infty} T_{L,i}^{\dagger}
\]
where
\[
(\overline{T}^v_L)^* (\overline{T}^v_L) = I
\]
for all \(i > 0\) \(\text{(35)}\)

In Eq. (35), the products that ultimately form \(\mathbf{U}_r^v\) and \(\mathbf{U}_l^v\) are developed by premultiplication. The zeroth transformation \((\overline{T}^v_L)^* N \overline{T}^v_L\) is chosen such that \((\overline{T}^v_L)^* N \overline{T}^v_L = I\).

It can be shown that provided each individual transformation \((\overline{T}^v_L)^* \) for \(i \geq 2\) has the orthogonality-like property given in Eq. (35) then \(\mathbf{U}_r^v\) and \(\mathbf{U}_l^v\) must implicitly obey \((\overline{T}^v_L)^* N \overline{T}^v_L = I\).

**V. Time Domain**

The solution for the characteristic roots of a second-order system and its associated vectors provides some motivation for utilizing a Clifford algebraic approach for second-order systems. We find re-inforcement of this motivation when we examine these systems in the time domain. This section identifies some of the properties exhibited by the state-space representations of self-adjoint first-order systems and extends these to second-order systems using Clifford algebra.

Consider a first-order system in which the equations of motion do not involve any mass,
\[
Kq + Dq = Q
\]
\(\text{(36)}\)

Suppose that forces may be applied at a limited number of terminal degrees of freedom and that displacements can be measured there. A selection matrix \(S\) indicates the terminals. The vector of forces at the terminals is \(T\), and the vector of displacements at the terminals is \(t\). Note that the vector \(T\) should not be confused with coordinate transformation matrices \(\{T_l\}, \{T_r\}\). For the principle of virtual work to apply, it is sensible to assert that if \(S\) relates \(t\) to \(q\), it must also relate \(Q\) to \(T\), and so
\[
t = S^T q, \quad Q = ST
\]
\(\text{(37)}\)

Note that Eq. (37) is not strictly a coordinate transformation. Unlike Eq. (18) in which we applied the constraint \(q = Tq\) and then used vector \(r\) as a complete representation of system deflections, Eq. (37) implies that \(q\) remains the only complete representation of system deflections.

By setting this in state-space form, we can use \(q\) as the state vector, \(T\) as the vector of inputs, and \(t\) as the vector of outputs, and we obtain\[
q = Aq + Bt
\]
\(\text{with } A = -D^{-1}K, \quad B = D^{-1}S, \quad C = S^T\)
\(\text{(38)}\)

If we began with a self-adjoint system \((K = K^T \text{ and } D = D^T)\), then we must find that the state-space system is also self-adjoint, and this is manifested as
\[
(CA^L B) = (CA^L B)^T \quad \text{for all } L
\]
\(\text{(39)}\)

If we intended to perform extensive response computations for a known first-order system, we would invariably solve the generalized eigenvalue problem for \((K, D)\) to find a coordinate transformation such that the new \(A\) matrix was diagonal. Thus,
\[
U^T K U = \Lambda, \quad U^T D U = I, \quad q = Ux
\]
\(\text{(40)}\)

and then
\[
\dot{x} = Ax + BT
\]
\(t = Cx
\)
\(\text{with } A = -\Lambda, \quad B = U^T S, \quad C = S^T U
\]
\(\text{(41)}\)

The symmetry of this first-order system is then evident because the output matrix \(C\) is the transpose of the input matrix \(B\), and the \(A\) matrix is diagonal.

We now attempt to make the symmetry in a self-adjoint second-order system equally apparent. The second-order system of interest is represented by Eq. (1), and the same selection matrix \(S\) applies to this system relating the terminal forces and displacements to the complete vector of forces and displacements according to Eq. (37). Casting this in a form similar to the standard state-space form, we have
\[
\begin{pmatrix}
K & 0 \\
0 & M
\end{pmatrix}\begin{pmatrix}
\dot{q} \\
q
\end{pmatrix} = \begin{pmatrix}
K & 0 \\
0 & S
\end{pmatrix}\begin{pmatrix}
\dot{t} \\
t
\end{pmatrix}
\]
\(\text{(42)}\)

Applying the (second-order) coordinate transformations implicit in Eq. (17), we have
\[
\begin{pmatrix}
\dot{\bar{x}} \\
\dot{\bar{y}}
\end{pmatrix} = \begin{pmatrix}
0 & \Omega \\
-\Omega & 2\zeta\Omega
\end{pmatrix}\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} - \begin{pmatrix}
-Y^S S \\
Z^S \end{pmatrix} \bar{T}
\]
\(\text{(43)}\)

In Eq. (43), the subscripts have been dropped from \((W, X, Y, Z)\) because the symmetry of the system means that \(W_r = W_l = W, \ X_r = X_l = X, \ Y_r = X_l = Y, \) and \(Z_r = Z_l = Z\). Differentiating a second time, we find
\[
\begin{pmatrix}
\ddot{\bar{x}} \\
\ddot{\bar{y}}
\end{pmatrix} = \begin{pmatrix}
0 & \Omega \\
-\Omega & 2\zeta\Omega
\end{pmatrix}\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} - \begin{pmatrix}
-Y^S S \\
Z^S \end{pmatrix} \bar{T}
\]
\(\Rightarrow \begin{pmatrix}
\ddot{\bar{x}} \\
\ddot{\bar{y}}
\end{pmatrix} = \begin{pmatrix}
0 & 2\zeta\Omega \\
-2\zeta\Omega & 2\zeta^2\Omega - \zeta^2\Omega^2
\end{pmatrix}\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} - \begin{pmatrix}
-Y^S S \\
Z^S \end{pmatrix} \bar{T}
\]
\(\text{(44)}\)

The final step in Eq. (44) requires that we utilize Eq. (17). From Eq. (17b) (for general unsymmetrical systems), it is elementary to observe that
\[
\left( \begin{pmatrix}
W_r \\
Y_r \\
Z_r
\end{pmatrix} \right)^{-1} = \begin{pmatrix}
W_l & -X_l \\
-Y_l & Z_l
\end{pmatrix} \begin{pmatrix}
K & 0 \\
0 & M
\end{pmatrix}
\]
\(\text{(45)}\)

Using this in conjunction with Eq. (17a), we find
\[
\begin{pmatrix}
W_l^T \\
Y_l^T \\
Z_l^T
\end{pmatrix} \begin{pmatrix}
0 & K \\
K & D
\end{pmatrix} = \begin{pmatrix}
0 & \Omega \\
\Omega & 2\zeta\Omega
\end{pmatrix}\begin{pmatrix}
W_l^T \\
Y_l^T \\
Z_l^T
\end{pmatrix} \begin{pmatrix}
K & 0 \\
0 & M
\end{pmatrix}
\]
\(\text{(46)}\)

We have implicitly assumed already that \(K\) is nonsingular. The first block column of Eq. (46) then provides the justification for the final step in Eq. (44):
\[
Y_l^T = -QX_l^T, \quad Z_l^T = (QW_l^T - 2\zeta\Omega X_l^T)
\]
\(\text{(47)}\)

We can show the same identities for the right vectors. Although for the present purposes we do not need to distinguish between \(W_r\) and \(W_l, X_r\) and \(X_l\), we shall find it useful later to maintain this.
In Eq. (48), x and y are parts of the state vector formed as linear combinations of the displacements and velocities. The state is fully known if both x and y are known. This equation is very similar in form to a state-space equation with the obvious difference being that the second derivative of the state vector is expressed rather than the first. Reusing the symbols A, B, and C to now represent the matrix quantities in Eq. (48), we have

\[
\begin{align*}
A &= \begin{pmatrix} -\Omega^2 & -Z\Omega^2 \\ 2Z\Omega^2 & (2Z\Omega^2 - 2\Omega^2) \end{pmatrix} \\
B &= \begin{pmatrix} 0 & -\Omega \\ -\Omega & 2Z\Omega \end{pmatrix} \\
C &= \begin{pmatrix} S^T Y & S^T Z \end{pmatrix}
\end{align*}
\]

(49)

The symmetry is not yet as obvious as it was in Eq. (41) for the first-order system. To see it, represent A, B, and C using matrices \( \hat{A} \), \( \hat{B} \), and \( \hat{C} \) of Clifford numbers whose dimensions are half of those of A, B, and C, respectively. In Sec. IV, we used the symbol \( \hat{O} \) to denote the Clifford representation matrix on the right side of Eq. (17a):

\[
\hat{A} = \hat{O}^\dagger \hat{A} \hat{O}, \quad \hat{B} = \hat{O}^\dagger \hat{B} \hat{O}^\dagger
\]

(50)

For every integer value of \( L \), we obtain

\[
(CA^L B) = C(\hat{O}^\dagger \hat{O})^L \hat{O}^\dagger \hat{C} \hat{C}^\dagger = \hat{C} \hat{O}^{2L} (\hat{O}^\dagger \hat{O})^L \hat{C}^\dagger
\]

(51)

Because \( \hat{O} \) is diagonal, we can find the following second-order equivalent of Eq. (41),

\[
(CA^L B) = (\hat{C} \hat{A} \hat{B})^\dagger
\]

for all \( L \)

(52)

We can use Eq. (48) to provide an interpretation of the second-order coordinate transformations discussed in Sec. III. Recall that in that section, we observed that, given any \( 2N \times 2N \) matrices \( \{P_{rs}, N_{rs}\} \) defined in Eq. (19), any square full-rank transformation matrices \( \{T_r\} \) could be applied to produce new system matrices \( \{P_{rs}, N_{rs}\} \). The relationship between the original state-vector \( \{\mathbf{q}, \mathbf{q}'\} \) and the modified state vector \( \{\mathbf{q}' \mathbf{q}'\} \) was given by Eq. (21), but we deliberately did not give a relationship between the force vectors. We can now develop this relationship.

Before continuing, we recycle some notation. Suppose that the selection matrix \( S \) as used earlier is the identity matrix. That is, all degrees of freedom of the model are accessible for the measurement of force/velocity and the application of force. Then \( \{\mathbf{f}, \mathbf{f}'\} \) is identical to \( \{\mathbf{q}, \mathbf{q}'\} \). To be consistent with the displacement-force parities \( \{\mathbf{q}, \mathbf{q}'\} \) and \( \{\mathbf{r}, \mathbf{r}'\} \), we now reuse \( S \) to be a (force/force-rate)-type vector related to the (deflection/deflection-rate)-type vector \( s \). Setting the selection matrix to the identity leads from Eq. (48) to

\[
\begin{align*}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} -\Omega^2 & -2Z\Omega^2 \\ 2Z\Omega^2 & (2Z\Omega^2 - 2\Omega^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
- \begin{pmatrix} 0 & -\Omega \\ -\Omega & 2Z\Omega \end{pmatrix} \begin{pmatrix} W^T \\ X^T \end{pmatrix} \begin{pmatrix} y \end{pmatrix} \\
\begin{pmatrix} q \\ \dot{q} \end{pmatrix} &= \begin{pmatrix} W_r & X_r \\ Y_r & Z_r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{pmatrix}
\end{align*}
\]

(53a)

In Eq. (53), \( x \) and \( y \) are in the nature of modal state vectors. Consider that we wish to apply the coordinate transformation of Eq. (21) to Eq. (53). This transformation implicitly requires that

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} W_r & X_r \\ Y_r & Z_r \end{pmatrix}^{-1} \begin{pmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}
\]

(54)

Substitute Eq. (54) into Eq. (53) and premultiply Eq. (53a) by a certain matrix to give

\[
\begin{pmatrix} \ddot{r} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} T_{l11} & T_{l12} \\ T_{l21} & T_{l22} \end{pmatrix} \begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix}^{-1} \begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} \\
- \begin{pmatrix} -\Omega^2 & -2Z\Omega^2 \\ 2Z\Omega^2 & (2Z\Omega^2 - 2\Omega^2) \end{pmatrix} \begin{pmatrix} W_r & X_r \\ Y_r & Z_r \end{pmatrix}^{-1} \begin{pmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \ddot{s} \end{pmatrix}
\]

(55)

provided that the premultiplying matrix is chosen to satisfy

\[
\begin{pmatrix} T_{l11} & T_{l12} \\ T_{l21} & T_{l22} \end{pmatrix} \begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix}^{-1} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]

We can use Eq. (17) to simplify Eqs. (55) and (56):

\[
\begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix}^{-1} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix}^{-1} \begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}
\]

(57a)

and

\[
\begin{pmatrix} W_L & -X_L \\ -Y_L & Z_L \end{pmatrix}^{-1} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} 0 & -K \\ -K & D \end{pmatrix}
\]

(57b)

Simplifying Eq. (55) using the preceding equations leads to

\[
\begin{pmatrix} \ddot{r} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} T_{l11} & T_{l12} \\ T_{l21} & T_{l22} \end{pmatrix} \begin{pmatrix} 0 & -K \\ -K & D \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}
\]

(58)

Equation (58) provides the required relationship between the original force vector and its rate of change \( \{\mathbf{q}, \mathbf{q}'\} \) and the generalized force vector for second-order systems:

\[
\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} T_{l11} & T_{l12} \\ T_{l21} & T_{l22} \end{pmatrix} \begin{pmatrix} 0 & -K \\ -K & D \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}
\]

(59)

It is clear from Eq. (58) that given the (constrained) system state \( \{r, s\} \) and sufficient information about the applied forcing \( \{Q, \dot{Q}\} \) we can conveniently determine the second derivative of (constrained) system state \( \{\ddot{r}, \ddot{s}\} \). The extension of Guyan reduction presented in Sec. III shows how the constraints on the system state can be chosen...
such that the first derivative of system state is exact when forces can be applied at only a subset of the model degrees of freedom.

VI. Restoring Second-Order Form

We noted earlier that if we begin with a system described by \( \{K, D, M\} \), we can form matrices \( P_{rr} \) and \( N_{uu} \) according to Eq. (19) and we can effect second-order coordinate transformations according to Eq. (20) using transformation matrices \( \{T_l, T_k\} \). When we depend on the choice of \( \{T_l, T_k\} \), the transformed representation of the system, \( P_{rr} \) and \( N_{uu} \), does not usually have the same form as the original \( P_{rr} \) and \( N_{uu} \). In particular, the extension of SEREP that we have outlined in Sec. II does not preserve the structure. If we are content to perform computations directly in \( C_l \), this is not an issue. However, there is an obvious attraction to being able to work directly with reduced system matrices \( \{K_{rr}, D_{rr}, M_{rr}\} \). Note that the aim of this section is to show that the process of restoring form is computationally possible. Any consideration of how to minimize loss of precision is deferred to another paper.

Recall these definitions of \( P \) and \( N \) from Eq. (19),

\[
P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}
\]

(60)

The conversion back to the structured case comprises finding a further second-order transformation after which \( P = P_{uu} \) and \( N = N_{uu} \).

If the system is not self-adjoint, this further transformation may comprise \( \{U_2, U_1\} \) with \( U_2 \neq U_1 \). If it is self-adjoint, we will endeavor to ensure that we retain its symmetry in \( \{P_{rr}, N_{uu}\} \). Either way, the further transformation will comprise three parts:

\[
U_L = U_{l1}U_{l2}U_{l3}, \quad U_R = U_{r1}U_{r2}U_{r3}
\]

(61)

Transformation \( \{U_{l1}, U_{r1}\} \) changes the system description from \( \{P_{rr}, N_{uu}\} \) to \( \{P_{l1}, N_{l1}\} \) with the result that \( N_{l1} \) equals the identity matrix.

Transformation \( \{U_{l2}, U_{r2}\} \) is selected so as to change the system description from \( \{P_{l1}, N_{l1}\} \) to \( \{P_{l2}, N_{l2}\} \) with the result that \( N_{l2} \) is equal to the identity matrix and \( P_{l1} \) has zeros in block (1,1).

Transformation \( \{U_{l3}, U_{r3}\} \) is developed so that it changes the system description from \( \{P_{l2}, N_{l2}\} \) to \( \{P_{uu}, N_{uu}\} \) with the result that blocks (1,2) and (2,1) of \( P_{uu} \) equal each other and block (1,1) of \( N_{uu} \).

A. Restoring Form for Systems Not Originally Symmetrical

If the original system is not self-adjoint, that is, any of its system matrices possesses some asymmetry, then we do not have to struggle to preserve any symmetry property in the similarity transformations. The first transformation can simply be expressed as

\[
U_{l1} = I, \quad U_{r1} = N_{l1}^{-1}
\]

(62)

Then \( N_{l1} \) is equal to the identity matrix. Because we wish also that \( N_{l2} \) equals the identity matrix, we must insist that \( U_{l2}, U_{r2} \) have this form:

\[
U_{l2}^T = \begin{pmatrix} A_L & 0 \\ 0 & B_L \end{pmatrix} \begin{pmatrix} I & X \\ Y & I \end{pmatrix}, \quad U_{r2} = \begin{pmatrix} I & X \\ Y & I \end{pmatrix} \begin{pmatrix} A_R & 0 \\ 0 & B_R \end{pmatrix}
\]

with \( A_L[I -X] \begin{pmatrix} I \\ Y \end{pmatrix}A_R = I \)

(63)

The requirement that block (1,1) of the new \( P \) matrix should comprise only zeros becomes

\[
(P_{11} + XP_{21}) + (P_{12} + XP_{22})Y = 0
\]

(64)

in which submatrices \( \{P_{11}, P_{12}, P_{21}, P_{22}\} \) are implicitly extracted from \( P_{uu} \) (not \( P_{rr} \)). There are many degrees of choice remaining in the selection of suitable \( U_{l2}, U_{r2} \). We would ordinarily set \( A_L = B_L = I \) and choose \( X \) such that \( (P_{12} + XP_{22}) \) is well conditioned. Then \( Y \) can be determined from Eq. (64) with ease.

The final transformation \( \{U_{l3}, U_{r3}\} \) is constructed as follows:

\[
U_{l3}^T = \begin{pmatrix} C_L & 0 \\ 0 & D_L \end{pmatrix}, \quad U_{r3} = \begin{pmatrix} C_R & 0 \\ 0 & D_R \end{pmatrix}
\]

(65)

with \( D_L P_{12} C_R = C_L P_{12} D_R = C_L C_R \). The submatrices \( P_{12} \) and \( P_{12} \) in this equation are extracted from \( P_{uu} \). Matrices \( C_L \) and \( C_R \) can be constructed arbitrarily, and \( D_L \) and \( D_R \) can be determined directly provided that both \( P_{11} \) and \( P_{12} \) are nonsingular.

After effecting this third transformation, we have a system \( \{P_{uu}, N_{uu}\} \) with the same form as \( \{P_{uu}, N_{uu}\} \) of Eq. (19).

B. Restoring Form for Systems Originally Symmetrical

If the original system is self-adjoint, then it is attractive to try to retain this symmetry, and this will implicitly mean that

\[
U_{l1} = U_{r1} = U_1, \quad U_{l2} = U_{r2} = U_2, \quad U_{l3} = U_{r3} = U_3
\]

(66)

The requirement for \( U_1 \) is then

\[
(U_1)^* N_{l1} U_1 = I
\]

(67)

Define the \( (2N \times 2N) \) matrix \( L \) as

\[
L = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

(68)

We can rewrite Eq. (67) as

\[
U_1^T L N_{l1} U_1 = L
\]

(69)

In the present circumstances we will always have symmetrical (\( LN_{l1} \)). Solution of the generalized eigenvalue problem \( eig((LN_{l1}) \) provides \( U_1 \).

Having ensured that \( N_{l1} \) equals the identity matrix, and being desirous that \( N_{l2} \) will be likewise, we must insist that \( U_2 \) is structured as follows:

\[
U_2 = \begin{pmatrix} I & Z \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \]

with \( E^T (I - ZZ^T) E = I \)

(70)

The additional requirement that block (1, 1) of \( P_{uu} \) should contain only zeros manifests itself as

\[
P_{11} + ZP_{21} + P_{12}Z^T + ZP_{22}Z^T = 0
\]

(71)

The blocks \( \{P_{11}, P_{12}, P_{21}, P_{22}\} \) are extracted from \( P_{uu} \) in this case. If we had a requirement that \( Z = Z^T \), Eq. (71) would translate into an algebraic Ricatti equation. However, we do not have this requirement, and solution of Eq. (71) is straightforward although it is not always possible to avoid determining complex \( Z \) (Ref. 17). Once \( Z \) has been computed, \( E \) and \( F \) can be found by eigenvalue-eigenvector decomposition of the matrices in Eq. (70).

The third transformation \( U_3 \) is constructed as follows:

\[
U_{r3} = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} \]

with \( H^T P_{21} G = G^T P_{12} H = G^T G \)

(72)

Submatrices \( P_{12} \) and \( P_{21} \) in this equation are extracted from \( P_{uu} \). Because symmetry automatically gives us \( P_{21} = P_{12} \), one of the identities in Eq. (72) is implicitly satisfied for all \( G \) and \( H \).

We can choose \( G \) arbitrarily as \( G = I \) and \( H \) follows from Eq. (72) directly. After effecting this third transformation, we have a system \( \{P_{uu}, N_{uu}\} \) with the same form as \( \{P_{uu}, N_{uu}\} \) of Eq. (19) and having only symmetrical matrices. We note, before closing, however, that in some cases, these matrices may be complex in which case the main purpose is not achieved. In such cases, it appears that symmetry must be sacrificed if a second-order system represented in the form \( \{K, D, M\} \) is required.

VII. Example

Several manipulations of the solution are expressed in Clifford formulation [Eq. (17)].

1) Characteristic solutions of the system are expressed in Clifford formulation [Eq. (17)].

2) The extension of Guyan reduction presented earlier is applied. The residuals in estimates of the full system receptance matrix are given for both conventional Guyan reduction and the extended method.
3) Transformations that convert the reduced system into tridiagonal form are noted. The system is a six-degree-of-freedom system whose mass matrix \( M_{qq} \) is the identity matrix and whose stiffness and damping matrices are

Each pair of elements represents a pair of complex roots, and these pairs have been ordered according to absolute values. Note that the last elements represent a pair of real roots. Because this system has symmetrical matrices only the right modal data need to be given. These modal data in Clifford form [refer to Eq. (17)] are

\[
W_R = 10^{-2} \times \begin{bmatrix}
6.2683 & -0.9811 & -0.7742 & -0.6492 & 0.1839 & -7.5231 \\
7.1474 & 2.9873 & 1.8744 & -0.2899 & -0.0353 & -2.5134 \\
6.4070 & 0.8597 & -1.8854 & 1.8361 & -0.0233 & -0.3626 \\
5.2257 & -1.6444 & -0.9221 & -1.6070 & 0.2674 & 0.5882 \\
3.2855 & -2.3345 & 1.5380 & 0.2623 & -1.2232 & 0.5197 \\
1.5347 & -1.3949 & 1.4864 & 0.8500 & 1.3269 & 0.2439
\end{bmatrix}
\]

(75)

\[
X_R = 10^{-3} \times \begin{bmatrix}
6.2683 & -0.9811 & -0.7742 & -0.6492 & 0.1839 & -7.5231 \\
7.1474 & 2.9873 & 1.8744 & -0.2899 & -0.0353 & -2.5134 \\
6.4070 & 0.8597 & -1.8854 & 1.8361 & -0.0233 & -0.3626 \\
5.2257 & -1.6444 & -0.9221 & -1.6070 & 0.2674 & 0.5882 \\
3.2855 & -2.3345 & 1.5380 & 0.2623 & -1.2232 & 0.5197 \\
1.5347 & -1.3949 & 1.4864 & 0.8500 & 1.3269 & 0.2439
\end{bmatrix}
\]

(76)

\[
Y_R = 10^{-11} \times \begin{bmatrix}
6.2683 & -0.9811 & -0.7742 & -0.6492 & 0.1839 & -7.5231 \\
7.1474 & 2.9873 & 1.8744 & -0.2899 & -0.0353 & -2.5134 \\
6.4070 & 0.8597 & -1.8854 & 1.8361 & -0.0233 & -0.3626 \\
5.2257 & -1.6444 & -0.9221 & -1.6070 & 0.2674 & 0.5882 \\
3.2855 & -2.3345 & 1.5380 & 0.2623 & -1.2232 & 0.5197 \\
1.5347 & -1.3949 & 1.4864 & 0.8500 & 1.3269 & 0.2439
\end{bmatrix}
\]

(77)

\[
Z_R = 10^{-11} \times \begin{bmatrix}
6.2683 & -0.9811 & -0.7742 & -0.6492 & 0.1839 & -7.5231 \\
7.1474 & 2.9873 & 1.8744 & -0.2899 & -0.0353 & -2.5134 \\
6.4070 & 0.8597 & -1.8854 & 1.8361 & -0.0233 & -0.3626 \\
5.2257 & -1.6444 & -0.9221 & -1.6070 & 0.2674 & 0.5882 \\
3.2855 & -2.3345 & 1.5380 & 0.2623 & -1.2232 & 0.5197 \\
1.5347 & -1.3949 & 1.4864 & 0.8500 & 1.3269 & 0.2439
\end{bmatrix}
\]

(78)

B. Reduced System

Degrees of freedom 1, 2, 3, and 5 are selected as master degrees of freedom. Applying straightforward Guyan reduction leads to a reduced system having four degrees of freedom. The characteristic roots for this system (again in Clifford form) are found to be

\[
\Omega = \text{diag}[7.9397, 26.3792, 32.8644, 16.0826]
\]

(79)

\[
2\zeta \Omega = \text{diag}[1.7183, 13.1143, 18.4291, 45.0215]
\]

(80)

Applying the extension of Guyan reduction described in Sec. III leads to a different reduced system having four degrees of freedom. The stiffness and damping matrices are found to be identical to those determined using the normal Guyan reduction, but the mass matrix is different and the following characteristic information is computed:

\[
\Omega = \text{diag}[7.9373, 26.9039, 32.5532, 32.8677]
\]

\[
2\zeta \Omega = \text{diag}[1.5136, 13.2899, 18.4295, 183.8630]
\]

(81)

Just as it is normal practice to examine the roots of the slave system in conventional static reduction, we find the roots of the slave system here to be

\[
\Omega = \text{diag}[38.0789, 46.9042]
\]

(82)

\[
2\zeta \Omega = \text{diag}[43.0000, 9.0000]
\]

(83)

When the full receptance matrix of the system is reconstructed at various different angular frequencies, the errors arising from the two
C. Part Way Through the Solution for Characteristic Roots Using Clifford Algebra

The four-degree-of-freedom system resulting from conventional Guyan reduction\(^7\) in the preceding subsection is used as a basis for one final numerical exercise. In this, we employ second-order similarity transformations to transform from an initial system having the pure form of \(P\) and \(N\) from Eq. (19) and a new matrix pair \(\{P, N\}\) whose Clifford representations \(\{ P, N \}\) are triagonal. This is given to illustrate that the numerical processes that are required to compress the system data are well developed. The reader will be able to form the original \(P\) and \(N\) matrices, \(\{ P_i, N_i \}\), by computing the Guyan-reduced system matrices, \([K_i, D_i, M_i]\). It may then be verified that \(P_{ii} = T_{ii}^T P_i T_{ii}\) and \(N_{ii} = (T_{ii})^T N_i T_{ii} = I\).

The new \(N\) matrix, \(N_{ii}\), is deliberately made to equal the identity matrix. We present the other (Clifford-) tridiagonal matrix \(P_{ii}\) in its full (real) form so that the reader may most easily see its structure:

\[
P_{ii} = \begin{bmatrix}
0 & 0 & 0 & 7.3318 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -27.3300 & 25.8789 & 0 \\
0 & 0 & 1.1180 & 0 & -17.0456 & -36.4101 & 15.0986 \\
0 & 0 & 9.4600 & 21.0248 & 0 & -17.0456 & -36.4101 & 15.0986 \\
0 & 0 & 0 & 21.0248 & 0 & -17.0456 & -36.4101 & 15.0986 \\
0 & 0 & 0 & 0 & 0 & -27.3300 & 19.6356 & 0 \\
0 & 0 & 0 & 0 & 0 & 15.0986 & 32.8772 \\
0 & 0 & 0 & 0 & -27.3300 & 19.6356 & 0 & 0 \\
0 & 0 & 0 & 0 & 19.6356 & 181.466 & -5.2355 & 0 \\
0 & 0 & 0 & 0 & -5.2355 & 18.9205 & -28.9609 \\
0 & 0 & 0 & 0 & 0 & -28.9609 & 23.6883 & 0
\end{bmatrix}
\] (82)

\[
T_{ii} = \begin{bmatrix}
0.0823 & -0.03383 & -0.04531 & 0.01219 & 0 & 0 & 0 & -0.00021 & -0.02893 \\
0.07578 & 0.00976 & -0.01218 & -0.02703 & 0 & 0 & 0 & -0.00006 & -0.00788 \\
0.06207 & -0.00302 & 0.03631 & -0.00357 & 0 & 0 & 0 & 0.00017 & 0.02320 \\
0.02971 & 0.02149 & 0.00619 & 0.02161 & 0 & 0 & 0 & 0.00003 & 0.00404 \\
0 & 0 & 0.2529 & 0.00141 & 0 & 0 & 0 & 0.6073 & 1.6959 \\
0 & 0 & -0.5653 & -0.00317 & 0.5556 & -0.05957 & 0.4689 & 0.8874 & 0 \\
0 & 0 & -0.07624 & -0.00041 & 0.4551 & -0.5355 & -0.7311 & -0.1140 & 0 \\
0 & 0 & 0.4556 & 0.00253 & 0.2178 & -0.6926 & 0.4468 & 0.7092 & 0
\end{bmatrix}
\] (83)

VIII. Conclusions

This paper has shown that some important aspects of the dynamics of general second-order systems can be written very compactly using Clifford algebra \(C_2\). We have indicated a computational advantage in the computation of characteristic roots that arises through the preservation of symmetry in the problem and the avoidance of complex arithmetic. We have presented extensions of two of the most important model-reduction methods used for undamped systems. Although this can be exposed and implemented without recourse to \(C_2\), it is both prompted by the formulation in Clifford form and made more compact through the use of Clifford notation.

The extension of static reduction, in particular, into the arena of generally damped unsymmetrical second-order systems transpires to be charmingly simple and immediately amenable to implementation. A final contribution is that the paper shows that the ability to recognize symmetry in the state-space formulation of self-adjoint first-order systems extends easily to second order systems through the use of Clifford algebra. The Clifford algebraic approach is illustrated in several aspects on a simple structural example.

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