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A MEASURE OF NON-PROPORTIONAL DAMPING

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Proportional or modal damping is often used as a simplified approach to model the effect of damping in linear vibrational mechanical systems. However, there are cases in which a general viscous damping is needed to simulate the dynamic of the system with sufficient accuracy. The scope of this paper is to investigate the difference between proportional and general viscous damping models. In case of general viscous damping, the modal marix of the underlying general eigenvalue problem depends on an orthonormal matrix, which represents the phase between different degrees of freedom of the model. It will be shown that in the case of proportional damping this orthogonal matrix becomes the identity matrix, which enables a real-valued normalisation of the modal matrix. Consequently, this orthogonal matrix can serve as a measure of the difference between proportional and general viscous damping models. Applications of the concept are demonstrated by two simulation examples.

1. INTRODUCTION

The quality of a model of a vibrational mechanical system is essential for a wide range of applications as, for instance, the prediction of system behaviour, damage detection and system design. A spatially discretised model with n degrees of freedom (dof) has the underlying eigenvalue problem

$$MY_o\Omega^2 + CY_o\Omega + KY_o = 0 \tag{1}$$

with the symmetric and positive-definite (p.d.) matrix of inertia M (kg) and stiffness matrix K (kg/s^2), and with the symmetric and positive-semidefinite (p.s.d.) damping matrix C (kg/s), and with the modal matrices of the complex eigenvectors Y_o (m) and complex eigenfrequencies Ω (1/s). To simplify the following investigations the description (1) is transformed into the dimensionless form

$$A_2X\Lambda^2 + A_1X\Lambda + A_0X = 0 \tag{2}$$

by multiplication of equation (1) with the positive constant $1/y_0m_o\omega_o^2$ and defining

$$(A_2, A_1, A_0, X, \Lambda) := \left(\frac{1}{m_o}M, \frac{1}{m_o\omega_o}C, \frac{1}{m_o\omega_o^2}K, \frac{1}{y_o}Y, \frac{1}{\omega_o}\Omega\right). \tag{3}$$

The constants y_o (m), m_o (kg) and ω_o (1/s) should be chosen with regard to the numerics in order to reduce the effect of truncating and rounding errors.

Modelling the contribution of inertia and stiffness is, in general, simpler than the modelling of dissipative effects because the latter is a purely dynamic property that cannot be measured statically. A common simplification is to assume proportional or modal damping, which is insufficient in some cases (see, for instance, Ibrahim [1], Gawronski and

Sawicki [2] and Lallement and Inman [3]). An arbitrary proportional damping model is given by [4]

$$A_{1p} = \sum_{i=0}^{n-1} c_i A_2 (A_2^{-1} A_0)^i \tag{4}$$

where the real-valued coefficients c_i have to be estimated. Indeed equation (4) is the most general expression for a damping matrix which satisfies the equivalent commutativity condition [5]

$$A_1 A_2^{-1} A_0 = A_0 A_2^{-1} A_1. \tag{5}$$

Each of the equations (4) and (5) are equivalent to the condition that the real-valued modal matrix $X_o \in \mathbb{R}^{n \times n}$ of the underlying eigenvalue problem

$$A_0 X_o + A_2 X_o \Lambda_o^2 = 0 \tag{6}$$

of the undamped model also diagonalises the damping matrix. Using the normalisation

$$X_o^{\mathrm{T}} A_2 X_o = I_n \tag{7}$$

where the superscript T indicates transposition, the eigenvalue problem (6) yields

$$X_o^{\mathrm{T}} A_0 X_o = -\Lambda_o^2 \tag{8}$$

with the diagonal matrix Λ_0 containing the purely imaginary eigenvalues λ_{oi} of the undamped model. Note that the modal matrix X_o is still not unique. If $E = \mathrm{diag}(\varepsilon_i)_{i=1,\ldots,n}$, where $\varepsilon_i \in \{-1, 1\}$ then also the matrix $X_o E$ satisfies the eigenvalue problem equation (6) with the normalisation equation (7). For the following investigation, it is assumed that a complete basis of eigenvectors exists, i.e. X_o is of full rank. Moreover $X_o \to X_o E$ is chosen in such a way that $\det(X_o) > 0$, which means that the columns of X_o form a positive-oriented basis of the vector space $\mathbb{R}^n$.

Both equations (7) and (8) are equivalent to

$$A_2 = (X_o X_o^{\mathrm{T}})^{-1} \tag{9}$$

$$A_0 = -(X_o \Lambda_o^{-2} X_o^{\mathrm{T}})^{-1}. \tag{10}$$

Inserting these expressions for A_2 and A_0 into equation (4) yields

$$A_{1p} = \sum_{i=1}^{n} c_i X_o^{-\mathrm{T}} X_o^{-1} (-X_o X_o^{\mathrm{T}} X_o^{-\mathrm{T}} \Lambda_o^2 X_o^{-1})^i \tag{11}$$

$$= X_o^{-\mathrm{T}} \underbrace{\left(\sum_{i=1}^{n} (-1)^i c_i \Lambda_o^{2i} \right)}_{=:\Gamma} X_o^{-1} \tag{12}$$

with the diagonal matrix Γ of dimensionless modal damping values γ_i. X_o indeed diagonalises this damping matrix. The eigenvalues of the eigenvalue problem with proportional damping

$$A_2 X_o \Lambda_p^2 + A_{1p} X_o \Lambda_p + A_0 X_o = 0 \tag{13}$$

can be calculated from

$$\Lambda_p^2 + \Gamma \Lambda_p - \Lambda_o^2 = 0 \tag{14}$$

yielding

$$\lambda_{pi} = -\frac{\gamma_i}{2} \pm \lambda_{oi}\sqrt{1 + \left(\frac{\gamma_i}{2\lambda_{oi}}\right)^2}. \tag{15}$$

Note that the second term on the right-hand side is imaginary because λ_{oi} is imaginary. Obviously, the modal damping matrix can be written as

$$\Gamma = -2\mathrm{Re}\{\Lambda_p\}. \tag{16}$$

In case of a non-proportional general viscous damping the modal matrix cannot be normalised to be real-valued. A simultaneous diagonalisation of all three matrices is no longer possible. In Section 2, the difference between proportional and general viscous damping is explored in terms of the modal matrix. It is shown that the difference between both damping models can be reduced to an orthogonal matrix, which serves as a measure of 'distance' between proportional and general viscous damping. In Section 3 this measure is calculated for two simulation examples.

2. PROPORTIONAL AND GENERAL VISCOUS DAMPING MODELS

A common method to solve the eigenvalue problem (2) of a general viscously damped model is to rewrite the problem by doubling the order

$$\underbrace{\begin{bmatrix} A_0 & 0 \\ 0 & -A_2 \end{bmatrix}}_{=:B} \underbrace{\begin{bmatrix} X & X^* \\ X\Lambda & X^*\Lambda^* \end{bmatrix}}_{=:Q_X} + \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_2 & 0 \end{bmatrix}}_{=:A} Q_X \underbrace{\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{bmatrix}}_{D} = 0 \tag{17}$$

where the superscript $*$ indicates the complex conjugate. Note that the partition of the matrix Q_X implies the model of an underdamped system, i.e. the eigenvalues are complex and occur as conjugate pairs. It is assumed that the imaginary parts of Λ are positive. Using the normalisation

$$Q_X^{\mathrm{T}} A Q_X = I_{2n} \tag{18}$$

the eigenvalue problem is equivalent to the two equations

$$AQ_X Q_X^{\mathrm{T}} = I_{2n} \tag{19}$$

$$BQ_X D^{-1} Q_X^{\mathrm{T}} = -I_{2n}. \tag{20}$$

Note that the products of modal quantities on the left-hand sides of equations (19) and (20) are real-valued, i.e.

$$Q_X Q_X^{\mathrm{T}} = 2\mathrm{Re}\left\{\begin{bmatrix} XX^{\mathrm{T}} & X\Lambda X^{\mathrm{T}} \\ X\Lambda X^{\mathrm{T}} & X\Lambda^2 X^{\mathrm{T}} \end{bmatrix}\right\} \tag{21}$$

$$Q_X D^{-1} Q_X^{\mathrm{T}} = 2\mathrm{Re}\left\{\begin{bmatrix} X\Lambda^{-1}\mathrm{X}^{\mathrm{T}} & XX^{\mathrm{T}} \\ XX^{\mathrm{T}} & X\Lambda X^{\mathrm{T}} \end{bmatrix}\right\}. \tag{22}$$

Evaluating the products of the partitioned matrices on the left-hand sides of equations (19) and (20), and comparing the result with the right-hand side leads to the four equations [6]

$$0 = \mathrm{Re}\{XX^{\mathrm{T}}\} \tag{23}$$

$$A_2 = \tfrac{1}{2}(\mathrm{Re}\{X\Lambda X^{\mathrm{T}}\})^{-1} \tag{24}$$

$$A_1 = -2A_2\mathrm{Re}\{X\Lambda^2 X^{\mathrm{T}}\}A_2 \tag{25}$$

$$A_0 = -\tfrac{1}{2}(\mathrm{Re}\{X\Lambda^{-1}X^{\mathrm{T}}\})^{-1}. \tag{26}$$

Note that equation (23) is independent of the model matrices and represents a restriction on measureable quantities. In equations (24)–(26) the model matrices are expressed in terms of the modal quantities. These equations cannot represent a one-to-one mapping because the $3n(n+1)/2$ independent real parameters of the three symmetric matrices of inertia, damping and stiffness are expressed in terms of $2n(n+1)$ real parameters of the complex modal matrix and the complex eigenvalues. This misfit is corrected by equation (23) which therefore is called the *basis constraint* (see, for instance, Garvey *et al.*, [7]). The importance of the basis constraint for the successful experimental identification of the mass, damping and stiffness matrices has been emphasised by Jeong and Nagamatsu [8]. They have proved that the basis constraint is equivalent to zero initial impulse response displacement. Starek and Inman [9] also emphasised that the real and imaginary parts of the complex modal matrix are not independent.

Exploring equation (23) reveals

$$\mathrm{Re}\{X\}\mathrm{Re}\{X\}^{\mathrm{T}} = \mathrm{Im}\{X\}\,\mathrm{Im}\{X\}^{\mathrm{T}} \tag{27}$$

and since X is non-singular, i.e.

$$\operatorname{rank}\left(\begin{bmatrix}\mathrm{Re}\{X\}\\ \mathrm{Im}\{X\}\end{bmatrix}\right) = n \tag{28}$$

or equivalently

$$H := \mathrm{Re}\{X\}\,\mathrm{Re}\{X\}^{\mathrm{T}} + \mathrm{Im}\{X\}\,\mathrm{Im}\{X\}^{\mathrm{T}} \quad \text{non-singular} \tag{29}$$

one finds

$$H = 2\mathrm{Re}\{X\}\,\mathrm{Re}\{X\}^{\mathrm{T}} = 2\mathrm{Im}\{X\}\,\mathrm{Im}\{X\}^{\mathrm{T}}. \tag{30}$$

Hence, neither $\mathrm{Re}\{X\}$ nor $\mathrm{Im}\{X\}$ is singular and equation (27) is equivalent to

$$(\mathrm{Re}\{X\}^{-1}\,\mathrm{Im}\{X\})^{-1} = (\mathrm{Re}\{X\}^{-1}\,\mathrm{Im}\{X\})^{\mathrm{T}} =: -\,\Theta. \tag{31}$$

Obviously Θ is an orthonormal matrix, i.e. $\Theta\Theta^{\mathrm{T}} = I_n$. Consequently, the modal matrix can be written as

$$X = R(I_n - j\Theta) \tag{32}$$

where R is a real-valued non-singular matrix. Since the number of independent parameters of an orthonormal matrix is $n(n-1)/2$ the number of independent real parameters of the modal quantities now is $n^2 + n(n-1)/2 + 2n = 3n(n+1)/2$, which corresponds to the number of independent parameters represented by the three symmetric matrices A_2, A_1 and A_0. Note that R will depend on Θ and on Λ in general. It will be shown later that a proportional damping is equivalent to the limit case

$$X_p := \lim_{\Theta \to I_n} X = R_p(1 - j). \tag{33}$$

In this case, the modal matrix X_p can be normalised to become real-valued, i.e. there exists a diagonal matrix Υ such that

$$X_p \Upsilon_0 \in \mathbb{R}^{n\times n}. \tag{34}$$

This normalisation is, of course, not unique (see, for instance, Ibrahim and Sesteri [10] and, Balmès [11]). It can be made unique by imposing additional conditions as, for instance, to require consistency with the normalisation (7) of the modal matrix of the undamped model, i.e.

$$X_0 = X_p \Upsilon_0 \tag{35}$$

which is equivalent to a 'minimum-phase' normalisation (see Ibrahim and Sestieri [10]). From equations (9) and (24) one finds

$$X_0 X_0^{\mathrm{T}} = 2\mathrm{Re}\{X_p \Lambda_p X_p\} = 4R_p \mathrm{Im}\{\Lambda_p\} R_p^{\mathrm{T}} \tag{36}$$

whilst equation (35) reveals

$$X_0 X_0^{\mathrm{T}} = -2j R_p \Upsilon_0^2 R_p^{\mathrm{T}} \tag{37}$$

Comparing both results yields

$$\Upsilon_0^2 = 2j \mathrm{Im}\{\Lambda_p\} \tag{38}$$

or

$$\Upsilon_0 = (1+j)\mathrm{Im}\{\Lambda_p\}^{1/2}. \tag{39}$$

Hence, in the case of general viscous damping the modal matrix

$$Y := X\Upsilon \tag{40}$$

is used and the normalisation equation (18) is changed correspondingly to

$$Q_Y^{\mathrm{T}} A Q_Y = \begin{bmatrix} \Upsilon^{-2} & 0 \\ 0 & \Upsilon^{-2*} \end{bmatrix} \tag{41}$$

where the superscript $*$ denotes the conjugate and $\Upsilon := (1+j)\mathrm{Im}\{\Lambda\}^{1/2}$. Note that equations (23)–(26) remain unchanged, and that equation (23) is already incorporated into the expression for X by equation (32).

In the next section, the difference between proportional and general viscously damped model is explored in terms of the modal quantities. It turns out that for the model class discussed in this paper (see Introduction) the orthogonal matrix Θ serves as a measure between these types of damping.

2.1. DIFFERENCE BETWEEN PROPROTIONAL AND GENERAL VISCOUS DAMPING

The definition of a measure of non-proportional damping has already been attempted in serveral ways. Bellos and Inman [12] use the mean value of the distribution of a non-proportionality index over the entire frequency range. Because the non-proportionality index is defined in terms of the diagonal of the modal damping matrix $\tilde{A}_1 := X_0^{\mathrm{T}} A_1 X_0$ and in terms of the modal coupling, the off-diagonal part of the modal damping matrix is not taken into account. A different approach has been reported by Liang *et al.* [13]. They define proportional damping via the symmetry of the eigenmatrix $X\Lambda X^{-1}$. Tong *et al.* [14] define an index of non-proportionality $I := (k-1)/(k+1)$ where k is the condition number of the normalised ($\tilde{A}_{1ik}/\tilde{A}_{1ii}$) modal damping matrix.

Of course, there are several equivalent criteria for a damping to be proportional as, for instance,

1. the real-valued modal matrix X_0 of the undamped model diagonalises the damping matrix A_1;
2. for fixed k, the phase $\mathrm{Im}\{X_{ik}\}/\mathrm{Re}\{X_{ik}\}$ is constant for all $i = 1, \ldots, n$;
3. the real and imaginary parts of the kth eigenvector are linearly dependent;
4. the modal matrix X can be normalised to become real-valued.

The first point follows directly either from the general proportional expression (4) or from the equivalent commutativity condition (5). The second point is obviously an equivalent formulation of the third, and if either of the second or third point holds, then the fourth point is evidently true. However all these criteria can be summarised in a single theorem.

Theorem 1. *Given a general viscously damped model of a vibrational system described by equations* (24)–(26) *and* (32) *with a positive-definite matrix A_2 the damping is proportional if and only if $\Theta = I_n$.*

Proof. Obviously in case $\Theta = I_n$ the real and imaginary parts of each complex eigenvector are linearly dependent. Hence, there exists a normalisation such that the modal matrix can be normalised to become real-valued. The crucial point is that such a normalisation is also possible in case of a diagonal matrix Θ. Because of the orthogonality this diagonal matrix can consists of -1's. Let us assume that $\Theta = \mathrm{diag}(\varepsilon_i)_{i=1,\ldots,n}$, with $\varepsilon_i \in \{-1, 1\}$, and that at least one $\varepsilon_i = -1$. From equation (24) one finds with reference to equation (32)

$$A_2 = \tfrac{1}{2}(\mathrm{Re}\{X\Lambda X^{\mathrm{T}}\})^{-1} \tag{42}$$

$$= \tfrac{1}{2}(R\mathrm{Re}\{(I - j\Theta)^2\Lambda\}R^{\mathrm{T}})^{-1} \tag{43}$$

$$= \tfrac{1}{2}(R\mathrm{Re}(-2j\Theta\Lambda\}R^{\mathrm{T}})^{-1} \tag{44}$$

$$= \tfrac{1}{4}(R\Theta\mathrm{Im}\{\Lambda\}R^{\mathrm{T}})^{-1}. \tag{45}$$

Since the imaginary part of all eigenvalues are positive (from definition of Λ) the diagonal matrix

$$4R^{\mathrm{T}}A_2R = (\Theta\,\mathrm{Im}\{\Lambda\})^{-1} \tag{46}$$

contains at least one negative entry, which contradicts the positive definiteness of the matrix A_2. $\square$

As an immediate consequence of Theorem 1 the difference between a general viscously and a proportional damped model is given by the measure

$$p := \|\Theta - I_n\| \tag{47}$$

where $\| \ldots \|$ is the spectral matrix norm, i.e. the largest singular value. Note that $p \in [0, 2]$. Moreover, Theorem 1 reveals that already in the limit case of proportional damping the orthgonal matrix Θ is not arbitrary because A_2 has to be positive definite. To explore the restrictions on Θ in general, one can rewrite equations (24)–(26) incorporating equation (32), which yields

$$A_2 = \tfrac{1}{2}[RH(\Lambda)R^{\mathrm{T}}]^{-1} \tag{48}$$

$$A_1 = 2A_2RH(-\Lambda^2)R^{\mathrm{T}}A_2 \tag{49}$$

$$A = \tfrac{1}{2}[RH(-\Lambda^{-1})R^{\mathrm{T}}]^{-1} \tag{50}$$

where the real-valued and symmetric matrix H is defined in general for an arbitrary diagonal matrix Λ by

$$H(\Lambda) := \mathrm{Re}\{\Lambda\} - \Theta\mathrm{Re}\{\Lambda\}\Theta^{\mathrm{T}} + \mathrm{Im}\{\Lambda\}\Theta^{\mathrm{T}} + \Theta\mathrm{Im}\{\Lambda\}. \tag{51}$$

Now the properties of the model matrices A_i can be translated to conditions on the matrix H:

$$A_2 \text{ p.d.} \Leftrightarrow H(\Lambda)\,\text{p.d.} \tag{52}$$

$$A_1 \text{ p.s.d.} \Leftrightarrow H(-\Lambda^2)\,\text{p.s.d.} \tag{53}$$

$$A_0\,\text{p.d.} \Leftrightarrow H(-\Lambda^{-1})\,\text{p.d.} \tag{54}$$

which leads to restrictions on the orthogonal matrix Θ. Of course, there are other constraints as, for instance,

$$4A_0 - A_1 A_2^{-1} A_1 \text{ p.d.} \tag{55}$$

which ensure the eigenvalues to occur as conjugate pairs (see, for instance, Inman and Andry [15]). Another problem is to estimate Θ from incomplete data. Obviously, the complete complex modal matrix is needed to calculate Θ using equation (31). Practically, only a few components of some modes will be available from tests. These problems are subject of ongoing investigation and are beyond the scope of this paper.

In the next section examples are given to demonstrate the measure defined by equation (47).

3. EXAMPLES

As explained in the previous section, non-proportionality is a geometric property of a spatially discretised model and is based on the orthogonal matrix Θ. The following two examples have been chosen to demonstrate how non-proportional damping is related to Θ. In order to compare the measure p of non-proportionality defined in equation (47) alternative indicators have been calculated

$$ps := \| G - G^{\mathrm{T}} \|, \quad G := Y \Lambda Y^{-1} \tag{56}$$

$$px := \| \operatorname{Im}\{Y\} \| \tag{57}$$

where Y is defined by equation (40). Moreover, the difference between the ith damped eigen-frequency and the ith natural eigenfrequency is given by

$$f_i := \operatorname{Im}\{\lambda_{oi}\} / \operatorname{Im}\{\lambda_i\}. \tag{58}$$

3.1. 2-dof MODEL

The following 2-dof simulation model stems from Lallement and Inman [3]. The mass matrix is the identity matrix, i.e. $A_2 = I_2$, and the stiffness and damping matrices are defined by

$$A_0 := \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \tag{59}$$

$$A_1 := 0.01(A_2 + A_0) + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{60}$$

where the damping parameter c varies between 0 (proportional damping) and 1. In Fig. 1 the ratio f_i defined in equation (58) between the damped and the natural eigenfrequencies is plotted as a function of c, and in Fig. 2 the three indicators defined in equations (47), (56) and (57) are depicted. All the three indicators show the same monotonic trend. Note that whilst the first damped eigenfrequency is always lower than the natural eigenfrequency, the second damped eigenfrequency exceeds the natural eigenfrequency with increasing non-proportionality.

3.2. A 50-dof VIBRATOR CHAIN

A vibrator chain with 50 equal masses $m = 1$ and springs $k = 1$ is depicted in Fig. 3. At the 25 th mass a single damper z is attached. In Fig. 4 the indicators p (solid), ps (dotted) and px (dash-dotted) are shown as functions of $z \in [0, 1]$. In contrast to the previous example the

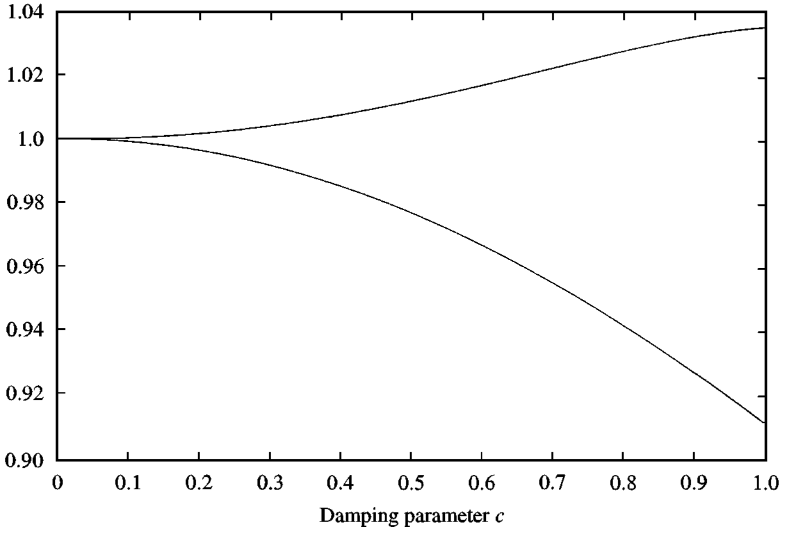

Figure 1. Ratio f_i, $i = 1, 2$, between natural and damped eigenfrequencies of the 2-dof model.

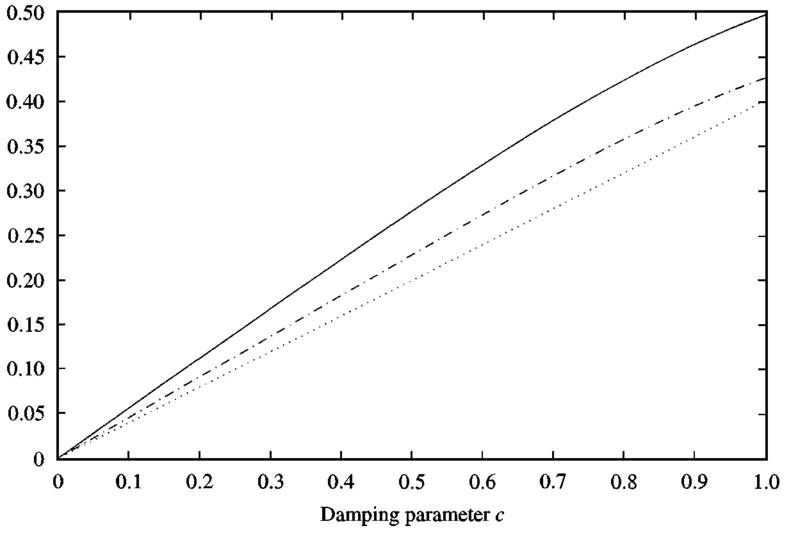

Figure 2. Indicators p (solid), ps (dashed) and px (dash-dotted) of non-proportionality of the 2-dof model.

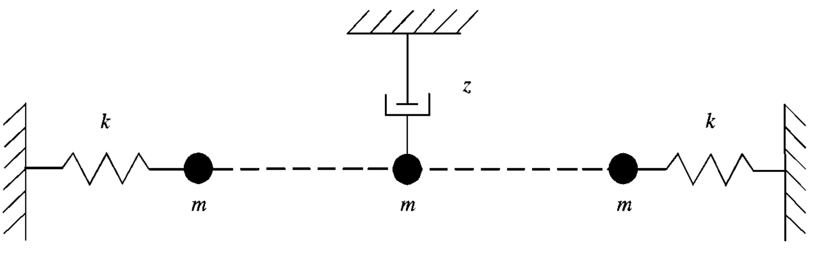

Figure 3. 50 dof vibrator chain with equal masses and equal springs and with one damper.

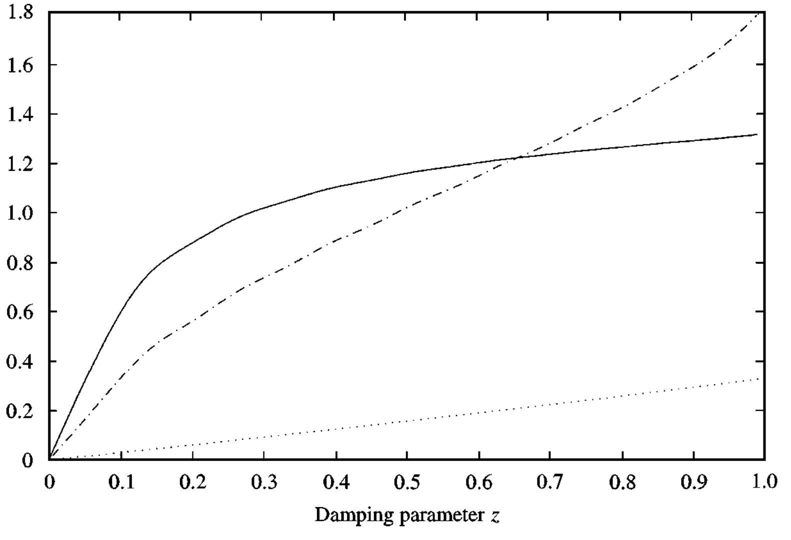

Figure 4. Indicators p (solid), ps (dashed) and px (dash-dotted) of non-proportionality of the 50-dof vibrator chain.

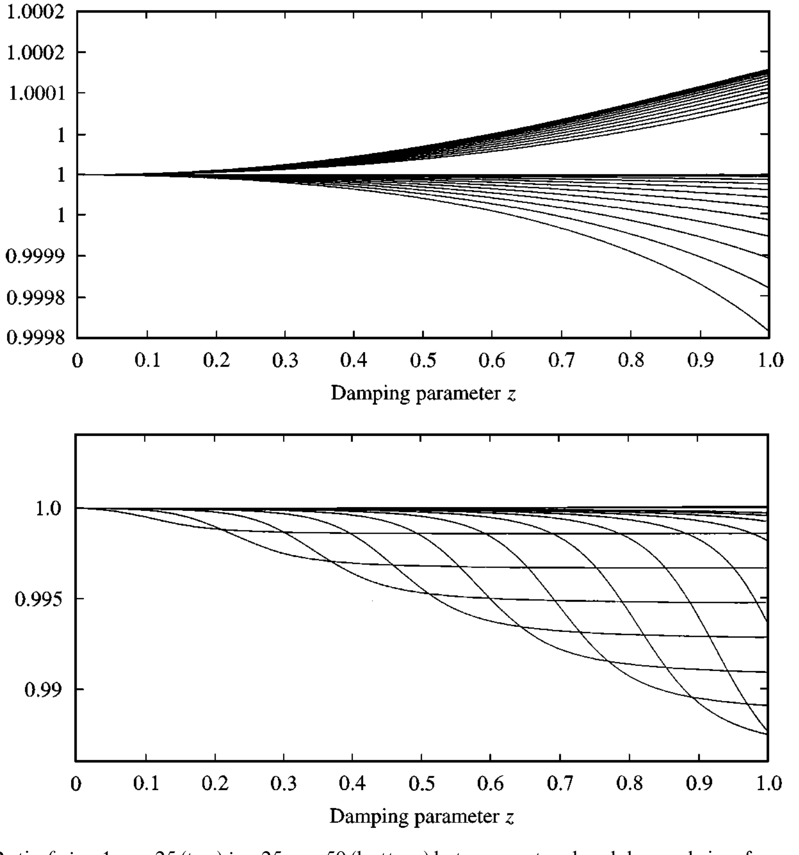

Figure 5. Ratio $f_i, i = 1, \ldots ,25$ (top) $i = 25, \ldots ,50$ (bottom) between natural and damped eigenfrequencies of the vibrator chain.

indicators show different trends. Whilst the indicator *ps* based on skew-symmetric part of the eigenmatrix and the indicator *px* based on the imaginary part of the modal matrix behave almost linearly and differ essentially by the inclination angles only, the indicator *p* based on the difference between the orthogonal matrix Θ and the identity matrix shows a distinct non-linear trend. The ratio between the natural eigenfrequencies and the damped eigenfrequencies is depicted in Fig. 5. The overall trend of the graphs (bottom) for f_i, $i = 26, \ldots, 50$ reveals the classical assumption, that an increase of the damping decreases the eigenfrequencies holds true, whereas a closer look at the lower frequency range f_i, $i = 1, \ldots, 25$ (top) shows a slight increase in the damped eigenfrequencies. Compared to the previous example the effect of damping on the eigenfrequencies is rather small. To clarify the effect of the non-proportional damping on the mode shapes, the first 8 modes are depicted in Figs. 6 and 7 for damping parameter $z = 1$. Because the vibrator chain is clamped at both ends the damper located in the midpoint of the chain affects the odd (Fig. 6) more than the even modes (Fig. 7). Indeed the real part (dash-dotted) of the eigenvectors Ye_i almost coincides with the eigenvectors $X_0 e_i$ of the undamped model (solid) for $i = 2, 4, 6, 8$, whilst for odd eigenvectors $i = 1, 3, 5, 7$ there is a rather small but obvious difference. Note that although the real part of the complex modal matrix Y is close to the real-valued modal matrix X_o the generalised damping matrix

$$\mathrm{Re}\{Y\}^{\mathrm{T}} A_1 \mathrm{Re}\{Y\} \tag{61}$$

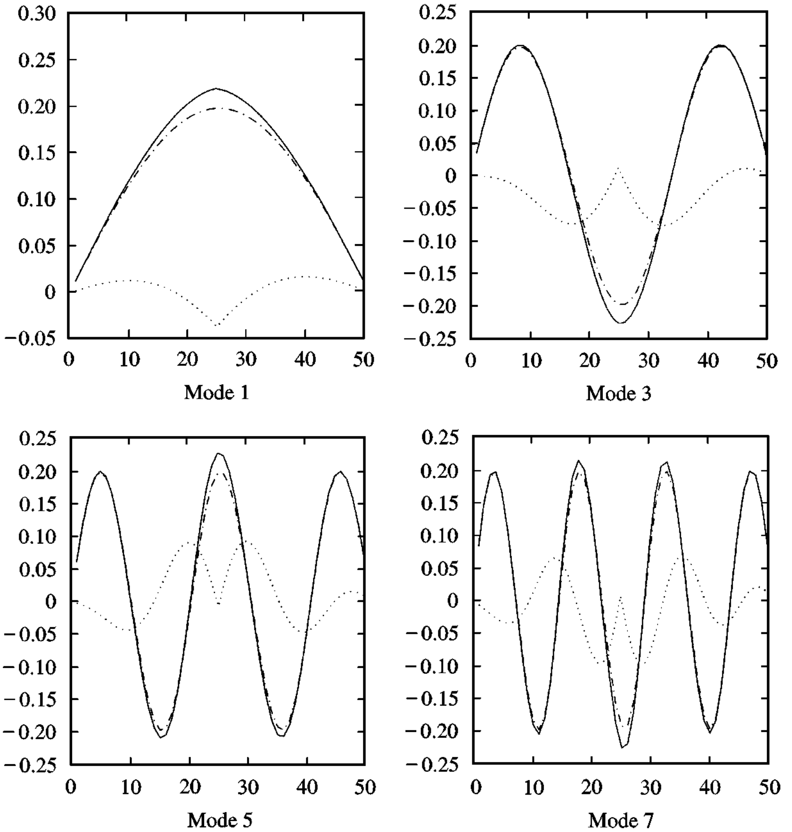

Figure 6. Mode shapes 1, 3, 5, 7 of the undamped model (solid) and the real (dash-dotted) and imaginary (dotted) parts of the damped model at $z = 1$.

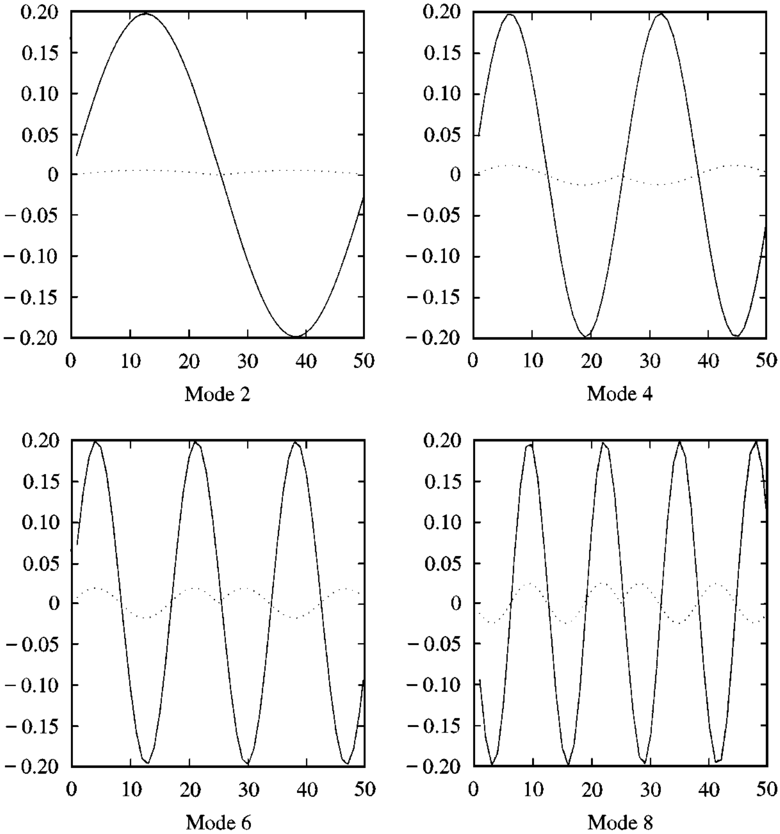

Figure 7. Mode shapes 2, 4, 6, 8 of the undamped model (solid) and the real (dash-dotted) and imaginary (dotted) parts of the damped model at $z = 1$.

is fully populated and is not diagonally dominant. The reason is the relative large difference between Θ and the identity matrix as the indicator $p \sim 1.3$ at $z = 1$ reveals. In Fig. 8 the diagonal and the first two upper and lower diagonals of Θ are depicted for the case $z = 1$. Whilst for the first dof Θ is close to the identity (only the second diagonals show some difference) this tendency is reversed for the last dof, as, for instance, dof 43–50 correspond to a rotation of about 90°.

4. CONCLUSION

The relation between general viscous damping and proportional damping has been explored in terms of the underlying general eigenvalue problem of linear models of elastomechanical systems. It has been shown that for non-defective models possessing a positive-definite mass matrix the difference between proportional and general viscous damping is equivalent to the difference between an orthonormal matrix and the identity matrix. This difference can serve as a measure of non-proportionality. It has been emphasised that this orthogonal matrix is not arbitrary because it is closely related to the positive definiteness of the model matrices. By two simulated examples this measure has been demonstrated and compared to alternative measures.

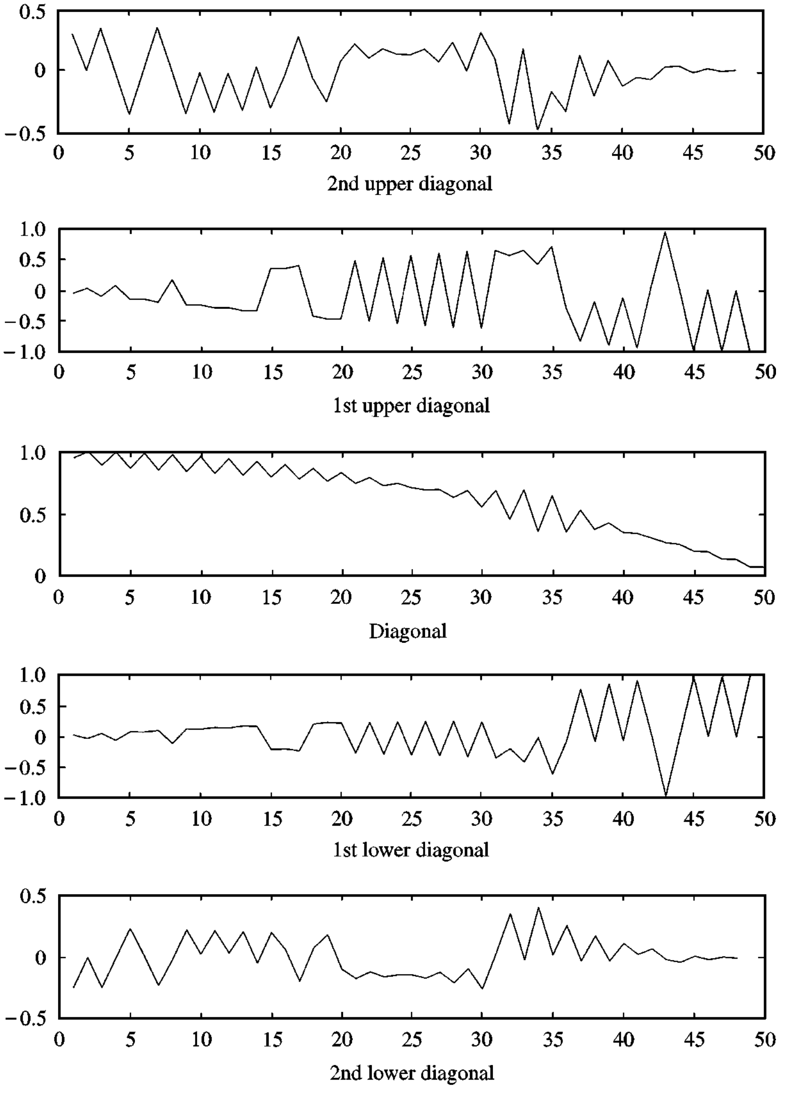

Figure 8. Super- and subdiagonal components of Θ at $z = 1$.

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