Partial and Segmented Modal Sensors for Beam Structures

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Abstract: The idea of using shaped sensors, and to a lesser extent actuators, for beam- and plate-type structures has been a subject of intense interest for many years. Typically for beam-type problems, the sensor width is varied along the length of the beam. The shape of the sensor is obtained by considering the modes of the beam and usually requires that the sensor covers the whole beam. This paper extends the previous results for distributed sensors for beams in two ways: the optimum sensor shapes that cover only a portion of the beam-type structures are derived and the optimum shapes for segmented sensors, that enable multiple modes to be sensed, are derived. The effect of geometric tolerances and the sensor output response to higher modes has been established. Although it is possible to derive the required sensor shapes, geometric tolerances will often make these sensors difficult to manufacture. The approach is demonstrated with a pinned-pinned beam example.

Key Words: Sensor, modal, vibration control

1. INTRODUCTION

The idea of using shaped sensors, and to a lesser extent actuators, for beam- and plate-type structures has been a subject of intense interest for many years. Using modal sensors in active control reduces problems of spillover, where high-frequency unmodeled modes affect the stability of the closed-loop system. For beam-type problems, a modal sensor may be obtained by varying the sensor width along the length of the beam. Often the sensor covers the whole beam, and the shape of the sensor may be derived from the equivalent mode of the beam using the mode shape orthogonality property. Lee and Moon (1990) and Clark and Burke (1996) provide good summaries of the state of the art in this area. Tanaka et al. (1996) placed a number of sensor patches on the structure to measure the response at a number of modes. They used sensors covering the whole beam and based their shape on combinations of the beam mode shapes. Zhuang and Baras (1994) discretised the sensor shape and used dynamic programming to optimise an energy function that describes the level of vibration in a composite beam.

Segmented sensors and actuators are often used in plate and shell applications. Usually the individual segment shapes are chosen a priori, and they are generally rectangular (Callahan and Baruh, 1995; Tzou et al., 1996). Shading of the sensors and actuators may be used to change their effectiveness over the area of the sensor or actuator (Burke and Hubbard, 1991).
This paper extends the previous results for distributed sensors for beams in two ways: the optimum sensor shapes that cover only a portion of the beam type structures are derived and the optimum shapes for segmented sensors, that enable multiple modes to be sensed, are derived. Although beams are used extensively throughout this paper, the results are easily extended to structures consisting of an assembly of beams, for example, space frames. All that is required is knowledge of the continuous mode shape in the region where the sensor will be applied. This may be obtained by using an exact model of the beams or approximated by using the element shape functions in a finite element model.

2. MODAL SENSORS FOR BEAMS

A brief review of modal sensors will now be provided, with emphasis on the use of sensors that cover only part of the beam. The papers by Lee and Moon (1990) and Clark and Burke (1996) should be consulted for a detailed analysis of the relationship of the sensor response to the beam equations of motion. This review will be followed by the derivation of sensor shapes that cover only part of the beam.

Suppose that a beam of length \( L \) has displacement mode shapes \( \phi_i(x) \), where \( x \) is the distance along the beam. The mode shapes will usually be normalised consistently, for example, they are often mass normalised. This definition implicitly includes the boundary conditions, as the mode shapes must satisfy any geometric boundary conditions. Also, it is not necessary for the beam to have a uniform cross section, but it must be possible to calculate the mode shapes either exactly or via an approximate method such as finite element analysis. If a numerical method is used, then the discretisation should be fine enough so that the mode shapes of interest are fully converged. The response of the beam, \( y(x, t) \), may then be written as a sum of the response in the modes, denoted by the modal participation factors \( p_i \), as

\[
y(x, t) = \sum_{i=1}^{\infty} p_i(t) \phi_i(x). \tag{1}
\]

Suppose a single PVDF sensor is placed on the beam with a shape defined by a variable width \( f(x) \), between \( x = a \) and \( x = b \). Incorporated into \( f(x) \) is both the physical width of the sensor and the polarisation profile of the material. The voltage output from this sensor is

\[
q(t) = K_v \int_a^b f(x) \frac{\partial^2 y(x, t)}{\partial x^2} dx, \tag{2}
\]

where the constant \( K_v \) is determined by the properties of the piezoelectric material and is

\[
K_v = \frac{E_p d_{31}(t_p + t_b)}{2}, \tag{3}
\]

where \( d_{31} \) is the relevant piezoelectric constant, \( E_p \) is the elastic modulus of the sensor, and \( t_p \) and \( t_b \) are the thickness of the sensor and the beam, respectively. The charge output may be decomposed into the contributions of the modes thus,
\[ q(t) = \sum_{i=1}^{\infty} q_i p_i(t), \]  

(4)

where

\[ q_i = K_i \int_{a}^{b} f(x) \frac{d^2 \phi_i(x)}{dx^2} dx \]  

(5)

is the charge output for the beam vibrating in the \(i\)th mode only. If it is desired to make the sensor sensitive to only the \(k\)th mode, and insensitive to the others, then the following equations must be satisfied:

\[ q_i = \delta_{ik}, \]  

(6)

where \(\delta_{ik}\) is the Kronecker delta. The sensitivity to the \(k\)th mode is given as unity for convenience, without loss of generality, as the sensor may be scaled to a convenient width, causing the output charge to be scaled similarly. The sensor may also be made sensitive to more than one mode, by requiring more than one of the \(q_i\) to be nonzero. For uniform beams, one solution that requires the sensor to cover the whole beam may be obtained from the orthogonality conditions as

\[ f(x) \propto EI(x) \frac{d^2 \phi_k(x)}{dx^2}, \quad a = 0, \quad b = L, \]  

(7)

where \(EI(x)\) is the flexural rigidity of the beam. Such a sensor will be sensitive to the \(k\)th mode and insensitive to all other modes. For a simply supported beam, the second derivative of the mode shape is proportional to the mode shape, and hence the sensor has the same shape as the corresponding mode. Of course, in using equation (7), some parts of the PDVF film sensor may have negative width, which may be interpreted as reversing the connecting wires for this part of the film. Since the mode shapes are smooth, going from a positive to a negative width will require passing through zero width, and so the parts of the sensor with positive and negative width will be physically disjoint, thus allowing the sensor to be wired as required. The alternative is to re-pole the PDVF (Lee and Moon, 1990).

If the PDVF does not cover the whole beam, then the calculation of an optimum shape is more involved. The method proposed in this paper is to use a complete set of basic functions over the part of the beam covered by the PDVF, \([a, b]\). The choice of these functions is considered in detail later. Over the interval \([a, b]\), the width of the PDVF may be written in terms of this orthonormal basis as,

\[ f(x) = \sum_{j=0}^{\infty} a_j \psi_j(x), \]  

(8)

where the constants \(a_j\) must be determined to ensure that the sensor is sensitive to a single mode, given by equation (6). Substituting equation (8) into equation (5),

\[ q_i = K_i \int_{a}^{b} f(x) \frac{d^2 \phi_i(x)}{dx^2} dx \]  

(5)
\[
\sum_{j=0}^{\infty} \beta_{ij} a_j = q_i, \quad i = 1, 2, \ldots,
\]  
(9)

where the constants \( \beta_{ij} \) are given by

\[
\beta_{ij} = \frac{1}{K_i} \int_{a}^{b} \frac{d^2 \phi_j(x)}{dx^2} \psi_j dx.
\]  
(10)

It may be thought that the series in equation (9) may be truncated and a solution obtained for the constants \( a_j \). If the orthonormal basis is chosen well, then the series for \( f(x) \) should converge quickly. Unfortunately, the resulting sensor shapes are very intricate and convergence is not readily obtained. This may be explained physically. As the number of terms in the truncated equation (9) is increased, the number of modes to which the sensor is made orthogonal is also increased. These higher modes have more intricate mode shapes, requiring a more intricate sensor shape.

The sensor is really required to be orthogonal to only a limited number of modes. In practice, the sensor output will be sampled, requiring an anti-aliasing filter that will limit the frequency range, and therefore the modes of interest. The resulting sensor and low-pass filter combination should be checked to make sure that the filter rolls off sufficiently quickly so that the response to the neglected high-frequency modes is small. The series for the sensor shape, equation (8), also needs to be truncated, but the number of terms from the orthogonal basis will generally be far greater than the number of modes retained. Extra conditions on the constants \( a_j \) are therefore needed to generate a solution. Conditions on the sensitivity do not work, because this would require increasing sensor complexity as the number of modes included is increased, as outlined above. A convenient condition is to minimise the curvature of the sensor shape, thus making the sensor easier to produce. Let the number of modes of interest be \( n \). The problem is posed as follows: obtain the \( m \) constants \( a_j, j = 0, \ldots, m - 1 \), such that the following objective function is minimised,

\[
J = \int_{a}^{b} \left( \frac{d^2 f(x)}{dx^2} \right)^2 dx.
\]  
(11)

subject to

\[
\sum_{j=0}^{m-1} \beta_{ij} a_j = q_i, \quad i = 1, \ldots, n
\]  
(12)

and where

\[
f(x) = \sum_{j=0}^{m-1} a_j \psi_j(x).
\]  
(13)

Note that the only difference between equations (8) and (9) and equations (12) and (13) is that the summations are truncated and only a limited number of modes are considered. The objective function is of the form,
where \( \gamma_{ij} = \int_a^b \frac{d^2 \psi_i(x)}{dx^2} \frac{d^2 \psi_j(x)}{dx^2} dx \). This minimisation problem may be easily solved using the Lagrange multiplier technique (Fletcher, 1981). Let \( \lambda_i \) be the Lagrange multiplier for the \( i \)th mode. The cost function is then augmented to

\[
J = \sum_{i,j=0}^{m-1} \gamma_{ij} \alpha_i \alpha_j + \sum_{i=1}^{m} \lambda_i \left( \sum_{j=0}^{m-1} \beta_{ij} \alpha_j - \delta_{ik} \right).
\]

The optimum values of the constants \( \alpha_i \), and also the Lagrange multipliers, is then obtained from

\[
\begin{bmatrix}
\beta_i \\
\gamma_i
\end{bmatrix} =
\begin{bmatrix}
\{\alpha\} \\
\{\lambda\}
\end{bmatrix}
= \begin{bmatrix}
\{q\} \\
\{0\}
\end{bmatrix},
\]

where,

\[
[\beta]_{ij} = \beta_{ij}, \quad [\gamma]_{ij} = 2\gamma_{ij}, \quad \{\alpha\} = a_j, \quad \{\lambda\} = \lambda_j, \quad \{q\} = q_i = \delta_{ik}.
\]

3. THE CHOICE OF BASIS FUNCTIONS

Thus far, no conditions have been placed on the choice of basis functions, except that they form a complete set. The choice of basis functions should not influence the sensor shape obtained. Choosing a good set of basis functions will, however, speed up convergence (as fewer terms in the series will be required) and eliminate ill-conditioning problems. Although the set of basis functions does not have to be orthogonal, using an orthogonal basis (Maddox, 1970) will usually help in the conditioning of the required calculations.

The functions \( \psi_j, j = 0, 1, 2, \ldots \), form an orthonormal basis over \([a, b]\), if,

\[
\int_a^b \psi_j(x) \psi_\ell(x) dx = \delta_{j\ell}, \quad j, \ell = 0, 1, 2, \ldots
\]

Although many orthonormal bases may be used, a convenient one is based on sines and cosines, through the Fourier series. In this case,

\[
\psi_j(x) = \begin{cases}
\frac{2}{b-a} \sin \frac{(j+1)\pi (x-a)}{b-a} & j \text{ odd} \\
\frac{2}{b-a} \cos \frac{j\pi (x-a)}{b-a} & j \text{ even} \\
\frac{1}{\sqrt{b-a}} & j = 0.
\end{cases}
\]
This basis also has the great advantage that the set of second derivatives of the basis functions is orthogonal, so that the matrix $\gamma$ is diagonal and

$$
\gamma_{jj} = \begin{cases} 
\frac{(j + 1)^4 \pi^4}{(b - a)^4} & j \text{ odd} \\
\frac{j^4 \pi^4}{(b - a)^4} & j \text{ even} \\
0 & j = 0.
\end{cases}
$$

(19)

Alternative sets of basis functions may be used but are not considered in detail in this paper. A polynomial set of basis functions may be derived using a Gram-Schmidt orthonormalisation process, although quoting the polynomials in closed form is more difficult. Other sets of orthogonal polynomials, for example, Legendre or Chebyshev polynomials, may be used. Walsh functions could also be used. It is unlikely that these orthogonal functions will have a great advantage over a Fourier series, and any computational advantage may quickly diminish once the effort of coding these functions into the method outlined in this paper is included. However, in some circumstances, for some sensor shapes, there may be significant advantages.

4. SENSITIVITY OF SENSOR OUTPUTS TO GEOMETRIC TOLERANCES AND HIGHER FREQUENCY MODES

The sensor designed by the method described thus far is sensitive to a single mode and insensitive to a chosen set of modes. Before such sensors can be implemented, it should be determined how accurately they must be manufactured. Also the response of the sensor to the neglected, usually high-frequency, modes must be evaluated. It is clear that if the shapes of sensors sensitive to different modes are similar, then it would be very difficult to produce a robust sensor. Of course, the determination of the sensor shape has relied on the mode shapes, normalised in some way. In practice, the output from the sensor will depend on the type of forcing applied, and particularly how the force is distributed among the modes, and also on the response of the sensor to each mode.

The response to the higher-frequency neglected modes may be derived using equation (5), and this may be compared to the unit output for the mode of interest. However, care must be exercised in interpreting these data, since in practice the higher modes may be excited less than the lower modes. The forcing may be incorporated by determining the response of the sensor to a white noise force input at a number of locations along the beam. The resulting receptance plots should show most of the sensor energy in the mode of interest. Suppose a force of magnitude $F(t)$ is applied at a position $x_j$ on the beam. Then, the part of the force applied to the $l$th mass normalised mode is given by $q_l \phi_l(x_j)$. The frequency response function between the force and the sensor response is then computed based on a sum of the contribution from the individual modes as

$$
H(\omega) = \sum_{l=1}^{\infty} \frac{q_l \phi_l(x_j)}{\omega_l^2 + 2j\omega_l\omega - \omega^2},
$$

(20)
where $\omega_i$ and $\zeta_i$ are the $i$th natural frequency and damping ratio, respectively, and where modal damping has been assumed. The $\phi_i(x_i)$ term in the numerator in equation (20) is generally of the same order of magnitude for all of the modes. The denominator generally increases as the natural frequency increases, thus making the contribution of the higher modes to the response smaller than the lower modes. The $i$th term in equation (20) has maximum magnitude of $\frac{q_i \phi_i(x_i)}{2 \zeta_i \omega_i^2 \sqrt{1 - \zeta_i^2}}$. In beam problems, the natural frequency is proportional to $i^2$, and hence the response in the $i$th mode decreases as $i^4$. We return to this phenomenon in the example, where the sensor outputs will be given per unit response in the $i$th mass normalised mode shape, but the above has demonstrated that the response in the higher modes will be small. Of course, if the frequency content of the force is concentrated in the higher modes, then this conclusion will not be valid. However, in the vibration control application, the response to broadband excitation and the transient response are the most important.

The sensitivity of the sensor output to geometric errors introduced by the cutting of the sensor may be obtained by looking at the worst effect on the output generated by geometric errors within a given bound. This will then give a direct comparison of the ability of a practical sensor to be sensitive to a single mode only. From equation (5), if the error on the sensor shape is $\epsilon(x)$, then the output generated by the $i$th mode is

$$e_i = K_s \int_a^b \epsilon(x) \frac{d^2 \phi_i(x)}{dx^2} dx.$$  

If the sensor is sensitive to the $k$th mode, then a convenient measure of the errors introduced is

$$E_k = \frac{1}{q_k} \sup_{|\epsilon(x)|<\epsilon_0} e_i = \frac{K_s \epsilon_0}{q_k} \int_a^b \left| \frac{d^2 \phi_i(x)}{dx^2} \right| dx,$$  

where $\epsilon_0$ is the assumed manufacturing tolerance, often given as a percentage of the maximum sensor width. Thus, the actual sensor shape is supposed to lie within a constant thickness band, given by $\epsilon_0$, around the nominal shape calculated by the method proposed in this paper. Since the sensitivity of the $k$th mode is designed to be unity, that is $q_k = 1$, the definition of equation (22) is valid irrespective of the mode of interest. Equation (22) gives a worst-case bound on the errors in the sensor output, and this bound will not be attained for all modes simultaneously.

5. SEGMENTED SENSORS FOR BEAMS

Separate sensors may be placed in separate parts of the beam to sense each mode in turn, but this may not be the most accurate or optimal method. It may be better to mix the output from the sensors in software. It is desired to make the sensors insensitive, or orthogonal, to the modes not required to be measured. Furthermore, it would be good to make the estimation of the modal responses from the sensor outputs as well conditioned as possible. A method to achieve this is now presented, and completely separate sensors will be a special case. The development will follow the dual sensor case closely and will consider measuring the
response of the first two modes only. The method is completely general, and the extension to
more modes or different modes is straightforward.

Suppose we have two PDVF sensors on the beam, the first covering the interval \([a_1, b_1]\)
with shape \(f_1(x)\), and the second covering the interval \([a_2, b_2]\) with shape \(f_2(x)\). It is not
necessary to make these intervals disjoint, although for practical applications this may be
desirable. The output from each sensor, when the beam is vibrating in its \(l\)th mode, is then

\[
q_{1l} = K_s \int_{a_1}^{b_1} f_1(x) \frac{d^2 \phi_l(x)}{dx^2} dx
\]

\[
q_{2l} = K_s \int_{a_2}^{b_2} f_2(x) \frac{d^2 \phi_l(x)}{dx^2} dx,
\]

where \(q_{1l}\) denotes the charge output of the first sensor to the response in the \(l\)th mode, and
similarly for the second sensor. We then require the output to be sensitive to only the first two
modes, so that

\[
q_{\ell i} = \begin{cases} 
\kappa_{\ell i} & \text{for } \ell, i = 1, 2 \\
0 & \text{otherwise} 
\end{cases}
\]  

(24)

The measured output of the sensors is the sum of the output for each mode. Thus, if
the beam is vibrating with magnitude (modal coordinate) \(p_1\) in the first mode and \(p_2\) in the
second mode, the output of sensor 1, \(\hat{q}_1\), and sensor 2, \(\hat{q}_2\), is

\[
\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{bmatrix} = [\kappa] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},
\]

(25)

where \([\kappa]_{ij} = \kappa_{ij}\). It is important to ensure that the response in the first two modes may be
estimated accurately or, equivalently, that equation (25) may be inverted to give the modal
response. Thus, the matrix \([\kappa]\) must be nonsingular but, more than that, should be well
conditioned with respect to inversion. Since this matrix may be chosen arbitrarily, a good
choice would be to make \([\kappa]\) orthogonal, which for the two-sensor case is conveniently
expressed as

\[
[\kappa] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}
\]

(26)

for any value of \(\theta\). If \(\theta = 0\), then sensor 1 measures mode 1 and sensor 2 measures mode
2. If \(\theta = 90^\circ\), then sensor 1 measures mode 2 and sensor 2 measures mode 1. If \(\theta = 45^\circ\),
then sensor 1 measures the sum of the modes and sensor 2 measures the difference.

We now return to designing the two sensors. As before, the shape of each of the sensor
segments may be expressed in terms of orthogonal basis functions. Indeed, we now have
to design two sensors in exactly the manner as Section 2, except that now each of the two
vectors \(\{q\}\) will contain two nonzero terms, given by equation (24), rather than the single
nonzero term in the standard single-sensor case. Obviously, as the number of sensors and the
number of modes of interest increase, the number of nonzero terms in this vector will also
increase.
6. A SIMPLY SUPPORTED UNIFORM BEAM EXAMPLE

As an example, consider a uniform, simply supported Euler-Benoulli beam, with mass normalised mode shapes

\[ \phi_i(x) = C_i \sin \frac{i \pi x}{L}, \]  

(27)

where \( C_i = \frac{\sqrt{3}}{\sqrt{\rho A L}} \) is the constant that ensures mass normalisation (Meirovitch, 1986), \( \rho \) is the (constant) mass density, \( A \) is the (constant) cross-sectional area, and \( L \) is the length of the beam. It is clear that the numerator in each term in equation (20) is of the same order, and hence the response at the higher modes reduces as \( i^2 \). All the following analysis has been performed using 61 basis functions, that is, \( m = 61 \), which is sufficient to obtain a converged solution for the sensor shape in the examples given below. Beams with other boundary conditions could be used, but this example is able to demonstrate all the features of the proposed approach.

6.1. Sensor Shapes

CASE 1: Suppose we chose to produce a sensor that is sensitive to each of the first four modes in turn, and insensitive to the four remaining lowest modes, that is, \( n = 5 \). If we assume that the whole beam is covered, that is, \( a = 0, b = L \), then Figure 1 shows the resulting sensor shapes. Remember that a negative value of sensor width requires the wiring to be reversed, and thus adjacent disjoint sensor segments will be connected in the opposite sense. In all figures of sensor shape, the widths are normalised to a maximum value of unity. It might be expected that the resulting shapes should be the mode shapes of the beam, and this is indeed the case for the second and fourth modes. The sensor shapes sensitive to the first and third modes are different because we have replaced the constraint that the modes are orthogonal to all the remaining beam modes (an infinite number of them) with a condition minimising the curvature of the sensor shape. The sensor shapes given in Figure 1 for the first and third modes do have a curvature that is less than that of the equivalent mode shape while still retaining the orthogonality to the other modes. Figure 2 shows the equivalent results when the orthogonality is required to the first 20 modes \( (n = 20) \). As expected, the sensors sensitive to the first and third modes are now closer to the corresponding mode shape.

CASE 2: Figure 3 shows the effect of reducing the length of the sensor so that \( a = 0.2L \) and \( b = 0.8L \). Once again, sensors are designed so that the sensor output is made sensitive to each of the first four modes in turn, and orthogonal to the remaining four lowest modes \( (n = 5) \). Clearly the sensor shapes now contain a contribution from all of the first five mode shapes. This is shown more clearly in Figure 4, where the number of modes considered has been increased to 10 \( (n = 10) \). As expected, as the sensors are required to become orthogonal to more modes, the geometric intricacy must increase.

CASE 3: A number of other sensors are designed covering a restricted region of the beam. In all cases, five modes are considered \( (n = 5) \) and the sensors to measure the response in each of the first four modes are designed in turn. Figures 5 through 7 show these results. What is clear is that the sensors designed for different modes can produce very similar shapes. This
Figure 1. Sensor shapes required to ensure sensitivity to the single given mode. \( n = 5, a = 0, b = L \). Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
Figure 2. Sensor shapes required to ensure sensitivity to the single given mode. \( n = 20, \ a = 0, \ b = L \). Also shown are the sensor outputs to each mode and the worst effect of a 5\% manufacturing tolerance.
Figure 3. Sensor shapes required to ensure sensitivity to the single given mode. $n = 5$, $a = 0.2L$, $b = 0.8L$. Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
Figure 4. Sensor shapes required to ensure sensitivity to the single given mode. $n = 10, a = 0.2L, b = 0.8L$. Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
Figure 5. Sensor shapes required to ensure sensitivity to the single given mode. \( n = 5, a = 0, b = 0.5L \).
Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
Figure 6. Sensor shapes required to ensure sensitivity to the single given mode. $n = 5$, $a = 0.2L$, $b = 0.6L$. Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
Figure 7. Sensor shapes required to ensure sensitivity to the single given mode. $n = 5$, $a = 0.3L$, $b = 0.7L$. Also shown are the sensor outputs to each mode and the worst effect of a 5% manufacturing tolerance.
is very unsatisfactory, as slight manufacturing tolerances are likely to lead to sensors that do not measure the response in a single mode. This problem is considered next.

6.2. Sensitivity to Neglected Modes and Geometric Tolerances

Also shown in Figures 1 through 7 are the sensor outputs to a unit response in the first 30 mass-normalised mode shapes. The bars in the plot show the response of a perfectly manufactured sensor. For example, in Figure 1 the sensor produces an output of unity for the mode that it is designed to be sensitive to and zero to the four modes that the sensor is designed to be orthogonal to. Although there is generally some response to the higher modes in Figure 1, the levels are generally small. The error bars on the plots show the effect of a geometric tolerance of 5% of the maximum half width of the sensor. Providing the sensor is manufactured within these bounds, the error bars give the maximum and minimum sensor output. Thus, the degradation in response to a given manufacturing tolerance may be easily assessed. Making the full-length sensor orthogonal to more modes makes the sensor closer to the standard orthogonal sensors and reduces the output at the higher modes (Figure 2). The degradation due to the manufacturing tolerances is similar to those in Figure 1.

Figures 3 through 7 show the sensor outputs and the effect of geometric tolerances for partial sensors. Clearly the results in all cases are much worse than those for a full-length sensor. It is clear from Figures 3 and 4 that making the sensor orthogonal to a large number of modes considerably increases the output at the higher modes and also makes the degradation due to manufacturing tolerances much worse. Figures 3 and 5 through 7 show that the position and length of the sensor also have a great influence; the sensor should cover as much of the beam as possible. Although the results seem quite bad and show that producing a robust sensor may be difficult, it should be remembered that the response at the higher modes is generally relatively small. As a demonstration, Figure 8 shows the absolute frequency response of the sensor designed for the first mode and given in Figure 3 (case 2, \( n = 5 \), \( a = 0.2L, b = 0.8L \)), to a white noise, unit force at \( x_f = 0.4L \), simulating the effect of an impulse force applied at this position. Damping of 1% has been added to each mode. Clearly the first mode response dominates and is 60 dB above any other peak. The response of a perfect sensor for the first mode is also given and shows that the only significant error is at high frequencies, where the response magnitude is low anyway.

6.3. Segmented Sensors

The segmented sensor is demonstrated by designing a dual sensor for the first two modes of the beam. The two sensors are assumed to cover almost all of the beam, and in this case \( \theta \) is chosen to be 45°. Figure 9 shows the shape of the sensors in this case. The sensor shapes are similar to those in Figure 5, and the robustness of this sensor to manufacturing tolerances is likely to be very poor.

7. CONCLUSIONS

This paper has extended the use of distributed modal sensors to measure the response of beams by considering sensors that cover only part of the beam. The effect of geometric
Figure 8. Response of a sensor designed for the first mode, \( n = 5, a = 0.2L, b = 0.8L \). The response magnitude is scaled by \( \sqrt{\rho AL} \), and the frequency by \( \sqrt{\frac{EI}{\rho AL^3}} \). Also shown (dashed) is the response of a perfect first mode sensor. The vertical dashed lines show the natural frequencies of the beam.

Figure 9. Segmented sensor shape required to ensure sensitivity to the first two modes. \( n = 5, a_1 = 0.01, b_1 = 0.49, a_2 = 0.51, b_2 = 0.99, \theta = 45^\circ \).
tolerances and the sensor output response to higher modes has been established. The use of segmented sensors has also been described. Although it is possible to produce the necessary sensor shapes, there are likely to be significant practical difficulties in their implementation. In particular, trying to apply modal sensors to a small part of the structure may lead to poor results, mainly because of the similarity of the sensor shapes required to measure different modes and the likely manufacturing tolerances. However, the methods proposed in this paper may be used to optimise the sensor position and size to produce as robust a sensor as possible.

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