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THE RELATIONSHIP BETWEEN THE REAL AND IMAGINARY PARTS OF COMPLEX MODES

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It is shown that a simple relationship exists between the real and imaginary parts of complex modes of all systems which can be represented by real and symmetric mass, stiffness and damping matrices. The relationship is most simply expressed in those cases where all roots are complex and where the real parts of all roots have the same sign. In these cases, the relationship can be expressed in a form where the imaginary part of the modal matrix is equal to the real part of the modal matrix post-multiplied by a matrix which involves an arbitrary real orthogonal matrix and some diagonal matrices which are determined directly from the complex roots. In other cases, there remains a relationship between the real and imaginary parts, but this must be expressed in a different way.

1. INTRODUCTION

In the analysis of an undamped system having N degrees of freedom and which is represented by real symmetrical stiffness and mass matrices $\mathbf{K}$ and $\mathbf{M}$, respectively, there are N real modes (in modal matrix $\mathbf{U}$) and N natural frequencies. It is not insisted here that the system matrices are positive definite but this would often be the case. The stiffness matrix of a free–free system has some zero eigenvalues and so it may be positive semi-definite. If the modes have been mass-normalized, then one can use the modal representation of the system in place of the system matrices and it is clear that there is no more compact representation than this in general since there are $N^2 + N$ independent parameters in $(\mathbf{K}, \mathbf{M})$ and the same number in the combination of modal matrix $\mathbf{U}$ and diagonal matrix $\mathbf{\Lambda}$ containing the eigenvalues.

If one allows the system to have proportional damping, then there will still be a real $(N \times N)$ modal matrix, $\mathbf{U}$, but there will now be $2N$ complex roots of the eigenvalue equation. There is a constraint that the complex roots occur in conjugate pairs so that one can be satisfied with holding only N real parts and N imaginary parts (as real numbers). The introduction of proportional damping accounts for only a further N real parameters in the specification of the system since proportional damping can be expressed by a diagonal matrix in the principal co-ordinates and one can regard the modal matrix, $\mathbf{U}$, as determined by $\mathbf{K}$ and $\mathbf{M}$. Thus, the initial number of independent real parameters is $N^2 + 2N$ in this case and the number of real parameters in the modal representation is $N^2 + 2N$.

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If one now allows the system to have non-proportional damping, (but retains the assumption of symmetry of all system matrices) then one begins with a space of $3N(N+1)/2$ independent real parameters and a modal representation can be found for this system which will have $2N$ complex modes and $2N$ complex roots to the eigenvalue equation. The $(N \times 2N)$ matrix of modes, $\mathbf{U}$ can be partitioned into two $(N \times N)$ parts as $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2]$ with $\mathbf{U}_2 = \text{conj}(\mathbf{U}_1)$ and each part has an associated set of N complex roots arranged in diagonal matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively with $\mathbf{S}_2 = \text{conj}(\mathbf{S}_1)$. Evidently, the modal representation can be compacted to $N^2 + N$ complex numbers by storing only $\mathbf{U}_1$ and $\mathbf{S}_1$. Alternatively, one might decide to store $\mathbf{U}_r(=(\mathbf{U}_1 + \mathbf{U}_2)/2)$ and $\mathbf{U}_i(=(\mathbf{U}_1 - \mathbf{U}_2)/2j)$ together with $\mathbf{S}_r(=(\mathbf{S}_1 + \mathbf{S}_2)/2)$ and $\mathbf{S}_i(=(\mathbf{S}_1 - \mathbf{S}_2)/2j)$ which collectively requires $2(N^2 + N)$ real numbers. Since there were fewer independent parameters than this in $(\mathbf{K}, \mathbf{C}, \mathbf{M})$, it is clear that one cannot in general select $\mathbf{U}_r$ and $\mathbf{U}_i$ arbitrarily and hope to reconstruct real, symmetrical $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ which will reproduce this set of complex modes. Indeed, if $\mathbf{U}_r$, $\mathbf{S}_r$ and $\mathbf{S}_i$ have been chosen, it is clear that there are only $N(N+1)/2$ independent parameters left in choosing $\mathbf{U}_i$. There are exactly $N(N+1)/2$ independent parameters in the determination of an $(N \times N)$ orthogonal matrix.

It is shown in what follows that (when all roots are complex and when all real parts have the same sign) the implicit constraints take the form . . . . $\mathbf{U}_i = \mathbf{U}_r(\mathbf{\Lambda}_y)^{-1}(\mathbf{\Lambda}_x - \mathbf{H} \cdot \mathbf{\Lambda}_z)$ where $\mathbf{H}$ is orthogonal and $\mathbf{\Lambda}_x$, $\mathbf{\Lambda}_y$, $\mathbf{\Lambda}_z$ are diagonal. All of the matrices in this equality are purely real. If $\mathbf{U}_i$ and $\mathbf{U}_r$ have been normalized in a certain way, then it is shown that $\mathbf{\Lambda}_x$, $\mathbf{\Lambda}_y$, $\mathbf{\Lambda}_z$ are related to the characteristic roots of the system.

This observation appears to be new in the literature. Since the general case of proportional damping was outlined by Caughey and O'Kelly [1] in 1965, there have been numerous publications on the area of complex modes ranging from the causes of these modes [2–4], the practicalities of deterimining the normal modes of $\mathbf{K}$, $\mathbf{M}$ from measured complex modes [5–9], the identification of the complex modes from frequency-response data [10] and the errors associated with ignoring the damping coupling between the "normal modes" [11–13].

Note, finally, that this paper deals only with systems represented by symmetrical matrices. It is almost certain that the concepts presented here extend to the more general case where the damping matrix, in particular, might have some component of skew-symmetry due to gyroscopic or Coriolis effects.

2. MODAL COMPUTATION OF THE RESPONSE OF THE GENERAL SELF-ADJOINT DAMPED SYSTEM

Let $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ be $(N \times N)$ matrices representing the stiffness, damping and mass properties of a system under a set of co-ordinates in which $\mathbf{q}$ is the vector of displacements and $\mathbf{Q}$ is the vector of forces. For a particular angular frequency, ω, the system behaviour is encapsulated in the equation

$$[\mathbf{K} + \mathrm{j}\omega\mathbf{C} - \omega^2\mathbf{M}]\mathbf{q} = \mathbf{Q}. \tag{1}$$

The system can be represented by determining the solution of the homogeneous equation. Because the system has damping, the characteristic roots will not be purely imaginary and in place of the relationship $\mathbf{q}(t) = \text{real}(\mathbf{q}\exp(\mathrm{j}\omega t))$ implicit in equation (1), one now uses $\mathbf{q}(t) = \text{real}(\mathbf{q}\exp(st))$. With $\mathbf{u}$ denoting any one of damped modes of the system and s the associated root, then equation (2) provides a definition of both the damped mode and its associated root:

$$[\mathbf{K} + s\mathbf{C} + s^2\mathbf{M}]\mathbf{U} = 0. \tag{2}$$

There are $2N$ distinct values of s which will make the determinant of $[\mathbf{K} + s\mathbf{C} + s^2\mathbf{M}]$ equal to zero. Since $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are purely real, the roots must occur in conjugate pairs and they can be gathered into two diagonal matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ with $\mathbf{S}_2 = \text{conj}\,(\mathbf{S}_1)$. If for each of the roots held on the diagonal of $\mathbf{S}_1$, the associated vector is inserted as a column of $\mathbf{U}_1$, then equation (2) becomes equation (3) below which shows that the solution vectors also occur in conjugate pairs and that $\mathbf{U}_2 = \text{conj}\,(\mathbf{U}_1)$:

$$\mathbf{K}\mathbf{U}_1 + \mathbf{C}\mathbf{U}_1\mathbf{S}_1 + \mathbf{M}\mathbf{U}_1\mathbf{S}_1^2 = 0 = \mathbf{K}\mathbf{U}_2 + \mathbf{C}\mathbf{U}_2\mathbf{S}_2 + \mathbf{M}\mathbf{U}_2\mathbf{S}_2^2. \quad (3)$$

The solutions of equation (2) can be determined by setting up a state–space form of the equation. This can take any one of a number of forms but for the present purposes, the following form is chosen:

$$\left[\begin{bmatrix} 0 & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} - s\begin{bmatrix} \mathbf{K} & 0 \\ 0 & -\mathbf{M} \end{bmatrix}\right] . \mathbf{v} = 0 = [\mathbf{A} - s\mathbf{B}] . \mathbf{v}. \quad (4)$$

The solution vectors $\mathbf{v}$ will each be composed as shown in equation (5) below and the roots, s, and vectors $\mathbf{v}$ can be arranged in matrices $\mathbf{S}$ and $\mathbf{V}$, respectively, as shown:

$$\mathbf{v}_i = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_i s_i \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \quad (5)$$

Notice that since $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are symmetric, then $\mathbf{A}$ and $\mathbf{B}$ are also symmetric and in the same way that $\mathbf{U}^T\mathbf{K}\mathbf{U}$ and $\mathbf{U}^T\mathbf{M}\mathbf{U}$ can be caused to be diagonal through correct choice of $\mathbf{U}$ (as the matrix of normal modes for the undamped system $\mathbf{K}$, $\mathbf{M}$), $\mathbf{V}^T\mathbf{A}\mathbf{V}$ and $\mathbf{V}^T\mathbf{B}\mathbf{V}$ are diagonal. With appropriate scaling of the complex modes,

$$\mathbf{V}^{\mathrm{T}}\mathbf{A}\mathbf{V} = \mathbf{S} \Rightarrow \begin{bmatrix} \mathbf{U}_1^{\mathrm{T}} & \mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} \\ \mathbf{U}_2^{\mathrm{T}} & \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix},$$

$$\mathbf{V}^{\mathrm{T}}\mathbf{B}\mathbf{V} = \mathbf{I} \Rightarrow \begin{bmatrix} \mathbf{U}_1^{\mathrm{T}} & \mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} \\ \mathbf{U}_2^{\mathrm{T}} & \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{K} & 0 \\ 0 & -\mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}. \quad (6)$$

Equations (6) can be used directly to establish a modal formula for the computation of the response of damped systems as an efficient alternative to $\mathbf{q} = [\mathbf{K} + \mathrm{j}\omega\mathbf{C} - \omega^2\mathbf{M}]^{-1}\mathbf{Q}$ when the response will be computed at a number of different frequencies:

$$\begin{bmatrix} \mathbf{q} \\ \mathbf{q}\mathrm{j}\omega \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2 \end{bmatrix}\left[\begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix} - \mathrm{j}\omega\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}\right]^{-1}\begin{bmatrix} \mathbf{U}_1^{\mathrm{T}} & \mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} \\ \mathbf{U}_2^{\mathrm{T}} & \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{Q} \end{bmatrix}$$

$$\Rightarrow \mathbf{q} = [\mathbf{U}_1[\mathbf{S}_1 - \mathrm{j}\omega\mathbf{I}]^{-1}\mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} + \mathbf{U}_2[\mathbf{S}_2 - \mathrm{j}\omega\mathbf{I}]^{-1}\mathbf{S}_2\mathbf{U}_2^{\mathrm{T}}]\mathbf{Q}. \quad (7)$$

The computation of response of a damped system is relevant to this paper only because it demonstrates that by measuring system response in the form of FRFs, it is perfectly possible to achieve the desired scaling of the complex modes. It has been noted by Ibrahim and Sestieri [5] that scaling of complex modes cannot be treated in quite the same arbitrary way as the scaling of real modes.

3. PROOF OF THE IDENTITY—CORRECTLY SCALED MODES

In this section, it is assumed that the modes have either been computed from a numerical model in which case it is straightforward to achieve the desired scaling of the modes or that they have been obtained from modal analysis based on measured FRFs and that they have been scaled so that equation (7) applies.

The proof then begins with the first of the two equations presented in equations (6). The central observation in the proof is that the $(2N \times 2N)$ matrix, $\mathbf{A}$, contains an $(N \times N)$ block of zeros. This equation can be post-multiplied by a certain matrix and pre-multiplied by its transpose to provide a direct reference to the block of zeros in $\mathbf{A}$ as follows:

$$[\mathbf{U}_1^{-\mathrm{T}}\mathbf{S}_1^{-1} \quad -\mathbf{U}_2^{-\mathrm{T}}\mathbf{S}_2^{-1}]\begin{bmatrix}\mathbf{U}_1^{\mathrm{T}} & \mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} \\ \mathbf{U}_2^{\mathrm{T}} & \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}0 & \mathbf{K} \\ \mathbf{K} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{S}_1^{-1}\mathbf{U}_1^{-1} \\ -\mathbf{S}_2^{-1}\mathbf{U}_2^{-1}\end{bmatrix}$$

$$= [\mathbf{X}^{\mathrm{T}} \quad 0]\begin{bmatrix}0 & \mathbf{K} \\ \mathbf{K} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{X} \\ 0\end{bmatrix} = 0,$$

$$\Rightarrow [\mathbf{U}_1^{-\mathrm{T}}\mathbf{S}_1^{-1} \quad -\mathbf{U}_1^{-\mathrm{T}}\mathbf{S}_1^{-1}]\begin{bmatrix}\mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{S}_1^{-1}\mathbf{U}_1^{-1} \\ -\mathbf{S}_2^{-1}\mathbf{U}_2^{-1}\end{bmatrix} = 0$$

$$\Rightarrow \mathbf{U}_1^{-\mathrm{T}}\mathbf{S}_1^{-1}\mathbf{U}_1^{-1} + \mathbf{U}_2^{-\mathrm{T}}\mathbf{S}_2^{-1}\mathbf{U}_2^{-1} = 0. \tag{8}$$

Note that in equations (8), the value of $\mathbf{X}$ is irrelevant. Upon observing that $\mathbf{U}_2 = \mathrm{conj}\,(\mathbf{U}_1)$ and $\mathbf{S}_2 = \mathrm{conj}\,(\mathbf{S}_1)$, it is evident that the last of equations (8) can be written more concisely as (9).

$$\mathrm{real}\,(\mathbf{U}_1^{-\mathrm{T}}\mathbf{S}_1^{-1}\mathbf{U}_1^{-1}) = 0. \tag{9}$$

One can now invoke the definitions of $\mathbf{U}_r$ and $\mathbf{U}_i$ and suppose that there is some relationship between these two matrices of the form $\mathbf{U}_i = \mathbf{U}_r \cdot \mathbf{Y}$. Note that $\mathbf{Y}$ is purely real (since $\mathbf{U}_i$ and $\mathbf{U}_r$ are themselves real) and assume that $\mathbf{U}_1$ and $\mathbf{U}_r$ are non-singular. This enables one to make an observation about $\mathbf{U}_1^{-1}$:

$$\mathbf{U}_1 = (\mathbf{U}_r + \mathrm{j}\mathbf{U}_i) = \mathbf{U}_r(\mathbf{I} + \mathrm{j}\mathbf{Y}) \Rightarrow \mathbf{U}_1^{-1} = (\mathbf{I} - \mathrm{j}\mathbf{Y})(\mathbf{I} + \mathbf{Y}^2)^{-1}\mathbf{U}_r^{-1}. \tag{10}$$

Matrix $\mathbf{U}_r^{-1}$ is necessarily full-rank. Equally, because $\mathbf{Y}$ is purely real, $\mathbf{Y}^2$ is positive definite and it can be shown that $(\mathbf{I} + \mathbf{Y}^2)$ must be invertible. Therefore no information is lost by recasting equation (9) as

$$\mathrm{real}\,((\mathbf{I} - \mathrm{j}\mathbf{Y})^{\mathrm{T}}\mathbf{S}_1^{-1}(\mathbf{I} - \mathrm{j}\mathbf{Y})) = 0. \tag{11}$$

Let $(\mathbf{S}_1)^{-1}$ be represented by $\mathbf{\Lambda}_1$ and let the real and imaginary parts of $\mathbf{\Lambda}_1 = (\mathbf{\Lambda}_r + \mathrm{j}\mathbf{\Lambda}_i)$ with $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_i$ being purely real. Since $\mathbf{S}_1$ and $\mathbf{S}_2$ are diagonal, then $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$, $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_I$ are also diagonal. The following is then obtained as an alternative form of equation (11):

$$[\mathbf{I} \quad \mathbf{Y}^{\mathrm{T}}]\begin{bmatrix}\mathbf{\Lambda}_r & \mathbf{\Lambda}_i \\ \mathbf{\Lambda}_i & -\mathbf{\Lambda}_r\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ \mathbf{Y}\end{bmatrix} = 0. \tag{12}$$

By completing the square and negating, equation (12) is made equivalent to

$$[\mathbf{I} \quad \mathbf{Y}^{\mathrm{T}}] \cdot \begin{bmatrix} \mathbf{\Lambda}_r & \mathbf{\Lambda}_i \\ \mathbf{\Lambda}_i & -\mathbf{\Lambda}_r \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{Y} \end{bmatrix} = \mathbf{\Lambda}_r + \mathbf{\Lambda}_i \cdot \mathbf{Y} + \mathbf{Y}^{\mathrm{T}} \cdot \mathbf{\Lambda}_i - \mathbf{Y}^{\mathrm{T}} \cdot \mathbf{\Lambda}_r \cdot \mathbf{Y} = 0$$

$$\Rightarrow (\mathbf{\Lambda}_r^{-0\cdot 5}\mathbf{\Lambda}_i - \mathbf{\Lambda}_r^{0\cdot 5} \cdot \mathbf{Y})^{\mathrm{T}}(\mathbf{\Lambda}_r^{-0\cdot 5}\mathbf{\Lambda}_i - \mathbf{\Lambda}_r^{0\cdot 5} \cdot \mathbf{Y}) - \mathbf{\Lambda}_r - \mathbf{\Lambda}_r^{-1}\mathbf{\Lambda}_i^2 = 0$$

$$\Rightarrow (\mathbf{\Lambda}_x - \mathbf{\Lambda}_y\mathbf{Y})^{\mathrm{T}}(\mathbf{\Lambda}_x - \mathbf{\Lambda}_y\mathbf{Y}) = \mathbf{\Lambda}_z^2, \tag{13}$$

where $\mathbf{\Lambda}_x = \mathbf{\Lambda}_r^{-0\cdot 5}\mathbf{\Lambda}_i$, $\mathbf{\Lambda}_y = \mathbf{\Lambda}_r^{-0\cdot 5}\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_z = \mathbf{\Lambda}_r^{-0\cdot 5}(\mathbf{\Lambda}_r^2 + \mathbf{\Lambda}_i^2)^{0\cdot 5}$. Any orthogonal matrix $\mathbf{H}$ (with $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I} = \mathbf{H}\mathbf{H}^{\mathrm{T}}$) can be used to determine a matrix $\mathbf{Y}$ which will satisfy equation (13):

$$(\mathbf{\Lambda}_x - \mathbf{\Lambda}_y\mathbf{Y}) = \mathbf{H} \cdot \mathbf{\Lambda}_z \Rightarrow \mathbf{Y} = (\mathbf{\Lambda}_y)^{-1}(\mathbf{\Lambda}_x - \mathbf{H} \cdot \mathbf{\Lambda}_z). \tag{14}$$

The identity has been proven but one additional note is worthwhile here. It appears that there has been an implicit assumption that $\mathbf{\Lambda}_r$ is positive since one is using $\mathbf{\Lambda}_r^{0\cdot 5}$ to relate purely real matrices. In general, with passive systems, one would ordinarily expect that all of the real parts of $\mathbf{S}_1$ and $\mathbf{S}_2$ would be negative (i.e., the system would be stable). It follows from this that $\mathbf{\Lambda}_r$ generally has all negative values on its diagonal. This case poses no difficulty in fact since equation (14) can be rearranged as

$$\mathbf{Y} = \mathbf{\Lambda}_r^{-1}\mathbf{\Lambda}_i - (\mathbf{\Lambda}_r)^{-0\cdot 5}\mathbf{H} \cdot (\mathbf{\Lambda}_r^2 + \mathbf{\Lambda}_i^2)(\mathbf{\Lambda}_r)^{-0\cdot 5}. \tag{15}$$

If $\mathbf{\Lambda}_r$ is negative definite, then $\mathbf{Y}$ as determined from equation (15) will be purely real. Some difficulty does arise when some of the entries of $\mathbf{\Lambda}_r$ are positive and some negative. Equally, some difficulty arises when some of the entries of $\mathbf{\Lambda}_r$ are zero. In these cases, one cannot construct the relationship between the real and imaginary parts as in equation (15) and must be content with using the relationship in the form (12).

4. PROOF OF THE IDENTITY—ARBITRARILY SCALED MODES

In section 2, it was noted that one can rely on achieving the particular scaling of the complex modes given in equation (6) if one has FRF information. Even in the case of measured data, a calibration error in any sensor would have the same effect across all modes and should not cause a problem to the constraint (since it was developed without any knowledge of the system matrices except that they were real and symmetrical).

However, sometimes, the modes of a system can be identified by observing the outputs from the system given a set of unknown white or coloured random inputs (as done, for example, by Cooper *et al*. [14]). In this case, the scale and the physical distribution of the input forces would be unknown and therefore, it would not be possible to ensure the desired scaling of the modes. One can, however, be assured that the modes thus determined will occur in complex conjugate pairs becasue the response comprises purely real numbers. One also recognizes that other workers will choose to scale their modes differently. Balmés [7] mentioned several possible scaling strategies. Thus, it is appropriate to look at whether the relationship between real and imaginary parts still holds when the modes have been scaled according to some other rule.

One begins these considerations by noting that, provided the modes remain in conjugate pairs, any arbitrary (possibly complex) scaling will result in an expression of the form

$$\begin{bmatrix} \mathbf{U}_1^{\mathrm{T}} & \mathbf{S}_1\mathbf{U}_1^{\mathrm{T}} \\ \mathbf{U}_2^{\mathrm{T}} & \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_1\mathbf{S}_1 & \mathbf{U}_2\mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1\mathbf{\Psi}_1\mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2\mathbf{\Psi}_2\mathbf{S}_2 \end{bmatrix}. \tag{16}$$

For clarity, note that with the original scaling, $\mathbf{\Psi}_1$ would be identical to $\mathbf{\Lambda}_1$. Then the development of the above section can be retraced exactly with $\mathbf{\Psi}_r$ and $\mathbf{\Psi}_i$ replacing $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_i$ from equations (12)–(14) to find that for some real orthogonal matrix, $\mathbf{H}$,

$$\mathbf{Y} = (\mathbf{\Psi}_r^{-1}\mathbf{\Psi}_i) - (\mathbf{\Psi}_r)^{-0\cdot5}\mathbf{H}\cdot(\mathbf{\Psi}_r^2 + \mathbf{\Psi}_i^2)(\mathbf{\Psi}_r)^{-0\cdot5}. \tag{17}$$

Without knowing $\mathbf{\Psi}_r$ and $\mathbf{\Psi}_i$, the usefulness of this identity initially appears curtailed. However, given an arbitrary $\mathbf{Y}$ matrix, of the form (17), one can extract all of the component parts $\mathbf{\Psi}_r$, $\mathbf{\Psi}_i$, and $\mathbf{H}$. This extraction would begin with determining what scaling factor, δ_i satisfied the condition that dividing row i by δ_i and multiplying column i by the same δ_i would cause the row and column to have equal norms. Then, recognizing that the norm of every row and column of $\mathbf{H}$ must be 1 provides a method for determining all unknowns.

5. A 4-DEGREE-OF-FREEDOM EXAMPLE

A ficticious 4-degree-of-freedom system is represented by the following system matrices where $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ represent stiffness, damping and mass respectively:

$$\mathbf{K} = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

By setting up and solving the resulting equation (4), one finds the complex roots and modes shown in Table 1, in which the real and imaginary parts of the various roots and modes are separated. As the roots (and modes) occur in complex pairs, one requires only to give the real and imaginary parts of one element of the pair and the other follows by simply negating the imaginary parts.

From this modal data, it is straightforward to compute real matrix $\mathbf{Y}$ which relates $\mathbf{U}_r$ and $\mathbf{U}_i$ (according to equation (10)). One finds the value for $\mathbf{Y}$ shown in Table 2. Using equation (13) to find the diagonal matrices $\mathbf{\Lambda}_x$, $\mathbf{\Lambda}_y$ and $\mathbf{\Lambda}_z$. and then applying equation (14), one can determine the unknown matrix $\mathbf{H}$ since one already has full knowledge of $\mathbf{Y}$. It is

TABLE 1

Complex modal data and complex roots for the 4-degree-of-freedom system

	Pair no. 1	Pair no. 2	Pair no. 3	Pair no. 4
Real parts of pair of roots	−0·064158315	−0·250985171	−0·041859052	−0·276330796
Imaginary parts of pair of roots	−1·256530540	−1·135885560	−0·818372708	−0·413311369
Real parts of modal vectors	0·421209338	−0·127040807	−0·082927194	0·196646053
	−0·082722431	−0·268357443	−0·140494226	0·535537593
	−0·005450471	0·208414779	−0·009028188	0·494961944
	−0·002105298	−0·055735339	0·487591987	0·279855967
Imaginary parts of modal vectors	0·061244473	0·174554056	0·018247309	0·037753076
	0·126669953	−0·049650601	0·019001072	0·195962647
	−0·064018227	0·017413550	0·100082235	0·155297273
	0·023125522	−0·067044888	0·034565787	−0·021250142

TABLE 2

The $\mathbf{Y}$ *matrix* ($\mathbf{Y} = \mathbf{U}_r^{-1} \cdot \mathbf{U}_i$)

0·00656652680	0·42515755509	0·03682099660	−0·08981119374
−0·39298787425	0·10435081954	0·17097116985	0·01459069006
−0·1805827756	−0·12014112160	0·01546335436	−0·21602818273
0·03587954858	−0·00626725158	0·13089817226	0·32026828681

shown in Table 3. It is trivial to check that this matrix $\mathbf{H}$ is orthogonal. Any error in the orthogonality of $\mathbf{H}$ can be measured by finding ($\mathbf{H}^{\mathrm{T}}\mathbf{H} - \mathbf{I}$).

6. POSSIBLE APPLICATIONS FOR THE IDENTITY

Three applications are immediately evident and a fourth is thought likely. In all cases, some additional work is necessary.

The identity provides the ability to check sets of complex modes determined from measured data and to force them to obey the constraint which must be present if the system from which these modes were obtained could be represented by real symmetrical mass, stiffness and damping matrices. The issues requiring further work relate primarily to the fact that there will normally be a difference between the number of measurement points and the number of pairs of modes identified, but additional work is also needed in the determination of some measure of the violation of the constraints and some method of modifying the modes with a minimum adjustment such that the constraint is obeyed. If the measured modal matrices are square, then the only knowledge required for computing such a check is available in the associated natural frequencies and damping levels. The example illustrates that a matrix $\mathbf{H}$ can be extracted from the scaled modal data and knowledge of the characteristic roots. If $\mathbf{H}$ is perfectly orthogonal, the constraints have been obeyed.

An alternative to equation (1) would be to build the identity implicitly into a modal-extraction algorithm for complex modes so that the modes produced implicitly satisfied the constraint. This might involve identifying $\mathbf{S}_r$, $\mathbf{S}_i$, $\mathbf{U}_r$ and $\mathbf{H}$ in which case some of the equations in the parameters would be second-order instead of linear. Additional work might succeed in recasting the constraint into a form in which such an identification process was broken into stages—perhaps determining $\mathbf{H}$ initally and then $\mathbf{U}_r$ once $\mathbf{H}$ is known. The identification problem might then be restored to comprising equations linear in the unknown parameters at every stage.

The identity provides for more compact storage of sets of complex modes and for more efficient calculation of the response of non-proportionally damped systems from the modal sets. The issues which will clearly need to be addressed include how to deal with the use of a small proportion of the modes and how to carry out some of the matrix multiplications which will inevitably involve a non-square orthogonal matrix $\mathbf{H}$.

TABLE 3

The $\mathbf{H}$ *matrix* ($\mathbf{H} = (\mathbf{\Lambda}_x - \mathbf{\Lambda}_y \cdot \mathbf{Y})\mathbf{\Lambda}^{-1}$)

0·99903383699	0·04288070983	0·00151662520	−0·00950458474
−0·04286905562	0·99896165737	0·01506453983	0·00330315360
−0·00114203388	−0·01502767878	0·99948434894	−0·02835314552
0·00960897435	−0·00331974519	0·02831599759	0·99954732313

It is conceivable that the identity could pave the way to a more efficient and more accurate solution method for the roots of the second order equation. If, for example, the **H** matrix could be determined through the solution of an $(N \times N)$ eigenvalue problem, then it is possible that the remainder of the problem could be solved with similar dispatch.

7. CONCLUSIONS

It is evident from a simple count of the number of independent real parameters in a set of symmetric matrices (**K**, **C**, **M**) and a count of the total number of real parameters in the usual complex modal representation of the same system that there are more parameters in the latter than there are in (**K**, **C**, **M**). Evidently, there are some implicit constraints on the parameters in the complex modal representation. These constraints manifest themselves as a necessary relationship between the real and imaginary parts of the complex modes of any second-order self-adjoint system and they have been written compactly and explicitly in this paper. It would appear that the implications of these constraints are both profound and several.

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REFERENCES

1. T. K. CAUGHEY and M M. J. O'KELLY 1965 *ASME Journal of Applied Mechanics* **32**, 583–588. Classical normal modes in damped linear dynamic systems.
2. M. IMREGUN and D. J. EWINS 1995 *Proceedings of the* 13*th International Modal Analysis Conference*, *Nashville*, *TN*, 496–506. Complex Modes—origins and limits.
3. Z. LIANG, M. TONG and G. C. LEE 1992 *International Journal of Analytical and Experimental Modal Analysis* **7**, 1–20. Complex modes in damped linear dynamic systems.
4. R. M. LIN, D. J. EWINS and M. K. LIM 1993 *International Journal of Analytical and Experimental Modal Analysis* **8**, 285–299. Identification of nonlinearity from analysis of complex modes.
5. S. R. IBRAHIM and A. SESTIERI 1995 *Proceedings of the* 13*th International Modal Analysis Conference*, *Nashville*, *TN*, 484–489. Existence and normalisation of complex modes in post experimental use in modal analysis.
6. A. SESTIERI and S. R. IBRAHIM 1994 *Journal of Sound and Vibration* **177**, 145–157. Analysis of errors and approximations in the use of modal co-ordinates.
7. E. BALMÉS 1997 *Mechanical Systems and Signal Processing* **11**, 229–243. New results on the identification of normal modes from experimental complex modes.
8. H. AHMADIAN, G. M. L. GLADWELL and F. ISMAIL 1995 *Proceedings of the* 13*th International Modal Analysis Conference*, *Nashville*, *TN*, 507–510. Extracting real modes from complex measured mode shapes.
9. M. IMREGUN and D. J. EWINS 1993 *Proceedings of the* 11*th International Modal Analysis Conference*. Realization of complex mode shapes.
10. W. LEURS, F. DEBLAUWE and F. LEMBREGTS 1993 *Proceedings of the* 11*th International Modal Analysis Conference*. Modal parameter estimation based on complex mode indicator functions.
11. H. FELSZEGHY 1993 *Transactions of the ASME Journal of Applied Mechanics* **60**, 456–463. On uncoupling and solving the equations of motion of vibrating linear discrete systems.
12. W. GAWRONSKI and J. T. SAVICKI 1997 *Journal of Sound and Vibration* **200**, 543–550. Response errors of non-proportionally lightly damped structures.
13. S. M. SHARUZ and P. A. SRIMATSYA 1997 *Journal of Sound and Vibration* **200**, 262–271. Approximate solutions of non-classically damped linear systems in normalised and physical coordinates.

14. J. Cooper, M. J. Desforges and J. Wright 1994 *Proceedings of the International Seminar on Modal Analysis* 1011–1023. Extraction of modal parameters from unknown coloured random input.
15. U. Prells. *Eine Regularisierungsmethode für die lineare Fehlerlokalisierung von Modellen elastomechanischer Systeme*. PhD Thesis, University of Hannover.

NOTE ADDED AT PROOF STAGE

At the time of writing, the authors were unaware of any previous work identifying the relationship between real land imaginary parts of complex modes. It has since emerged that this relationship has been noted for a different particular scaling of the modes [15] and a simpler form emerges in that case. The development in [15] does not generalize to arbitrarily scaled or real modes and it does not proceed via an equation in the form of (8) which offers useful additional generality.