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THE CONVERGENCE OF THE ITERATED IRS METHOD

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Static or Guyan reduction is widely used to reduce the number of degrees of freedom in a finite element model, but it is exact only at zero frequency. The Improved Reduced System (IRS) method makes some allowance for the inertia terms and produces a reduced model which more accurately estimates the modal model of the full system. The IRS method may be extended to produce an iterative algorithm for the reduction transformation. It has already been shown that this reduced model reproduces a subset of the modal model of the full system if the algorithm converges. In this paper it is proved that the iterated IRS method converges. It is also shown that the lower modes converge more quickly than the higher modes and that the master co-ordinates should be chosen to give an accurate static reduction.

1. INTRODUCTION

Model reduction, whereby the number of degrees of freedom in a model is reduced, is applied to large finite element models to give faster computation of the natural frequencies and mode shapes of a structure. Model reduction also has a role to play in experimental modal analysis since the reduced mass and stiffness matrices may also be used to compare the analytical and experimental models by using orthogonality checks. The transformation inherent in the model reduction schemes may also be used to expand the measured mode shapes to the full size of the finite element model, and these mode shapes may then be used in test analysis correlation or model updating exercises.

One of the oldest and most popular reduction methods is static or Guyan reduction [1]. In this process the inertia terms associated with the discarded degrees of freedom are neglected. However, whilst exact for a static model, when applied to a dynamic model the reduced model generated is not exact and often lacks the required accuracy. O'Callahan [2] proposed a modified method which he called the Improved Reduced System (IRS) method. In this approach an extra term is added to the static reduction transformation to make some allowance for the inertia forces. This extra term allows the modal vectors

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of interest in the full model to be approximated more accurately but relies on the statically reduced model.

The IRS method has been extended in two ways [3]: by using the transformation from dynamic reduction instead of static reduction as the basic transformation; and by introducing an iterative scheme where the corrective term is generated iteratively using the current best estimate of the reduced model. It has been demonstrated that the natural frequencies of the reduced model converge to those of the full model [3, 4]. In this paper a proof of this convergence is given.

It is not the purpose of this paper to consider the computational advantages of the iterated IRS method, as this has already been done [3, 4]. Even so, it is appropriate to make some comments in this regard. Determining whether the iterated IRS requires more computation than standard eigensolvers is extremely difficult, particularly since eigenvector extraction and the iterated IRS method are generally iterative procedures, and the number of iterations will depend on the properties of the full system and the choice of master degrees of freedom (in the case of IRS). For example, subspace iteration will require many iterations if the highest eigenvalue of interest is close to the lowest discarded eigenvalue. A poor choice of master degrees of freedom will require many iterations before the IRS method converges. The authors have shown that the iterated IRS method requires approximately the same number of floating point operations per iteration as subspace iteration [4]. Also, if the number of master degrees of freedom is much less than the number of slave degrees of freedom then iterated IRS requires, in total, approximately twice the number of floating point operations as Guyan reduction [3]. Dynamic condensation reduces the model about each resonance in turn and requires at least as much computational effort as Guyan reduction for each mode. Therefore dynamic condensation requires much more computational effort than the iterated IRS method.

In this paper, the standard IRS method is introduced in section 2 and in section 3 the IRS method is extended to include iteration. These two sections summarize the previous work on this algorithm [3]. In section 4 it is shown that if the iterated IRS method converges then the resulting transformation is the same as that obtained from the System Equivalent Reduction Expansion Process (SEREP). Then in section 5 it is proved that the iterated IRS method does indeed converge, and consideration is given to the factors, such as the selection of master degrees of freedom, that affect the speed of convergence. Some comments on the optimum selection of master degrees of freedom are given in section 6, before a numerical example is used to demonstrate the convergence properties of the method in section 7.

2. THE STANDARD IRS METHOD

In Guyan reduction [1], the deflection and force vectors, $\mathbf{x}$ and $\mathbf{f}$, and the mass and stiffness matrices, $\mathbf{M}$ and $\mathbf{K}$, are re-ordered and partitioned into separate quantities relating to master (retained) and slave (discarded) degrees of freedom. Upon assuming that no force is applied to the slave degrees of freedom and the damping is negligible, the equation of motion of the structure becomes

$$\left[\begin{array}{c|c} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \hline \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{array}\right]\left\{\begin{array}{c} \ddot{\mathbf{x}}_m \\ \hline \ddot{\mathbf{x}}_s \end{array}\right\} + \left[\begin{array}{c|c} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \hline \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{array}\right]\left\{\begin{array}{c} \mathbf{x}_m \\ \hline \mathbf{x}_s \end{array}\right\} = \left\{\begin{array}{c} \mathbf{f}_m \\ \hline \mathbf{0} \end{array}\right\}. \tag{1}$$

The subscripts m and s relate to the master and slave co-ordinates respectively. By neglecting the inertia terms in the second set of equations, the slave degrees of freedom may be eliminated so that

$$\left\{\begin{array}{c}\mathbf{x}_m \\ \hline \mathbf{x}_s\end{array}\right\} = \left[\begin{array}{c}\mathbf{I} \\ \hline -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\end{array}\right]\mathbf{x}_m = \mathbf{T}_s\mathbf{x}_m, \tag{2}$$

where $\mathbf{T}_s$ denotes the static transformation between the full state vector and the master co-ordinates. The reduced mass and stiffness matrices are then given by

$$\mathbf{M}_R = \mathbf{T}_s^{\mathrm{T}}\mathbf{M}\mathbf{T}_s, \qquad \mathbf{K}_R = \mathbf{T}_s^{\mathrm{T}}\mathbf{K}\mathbf{T}_s, \tag{3}$$

where $\mathbf{M}_R$ and $\mathbf{K}_R$ are the reduced mass and stiffness matrices. Note that any frequency response functions generated by the reduced matrices in equation (3) are exact only at zero frequency. As the excitation frequency increases the inertia terms neglected in equation (1) become more significant.

O'Callahan [2] improved the static reduction method by introducing a technique known as the Improved Reduced System (IRS) method. The method perturbs the transformation from the static case by including the inertia terms as pseudo-static forces. Obviously, it is impossible to emulate the behaviour of a full system with a reduced model and every reduction transformation sacrifices accuracy for speed in some way. O'Callahan's technique [2] results in a reduced system which matches the low frequency resonances of the full system better than static reduction. However, the IRS reduced stiffness matrix will be stiffer than the Guyan reduced matrix and the reduced mass matrix is less suitable for orthogonality checks than the reduced mass matrix from Guyan reduction [5].

The IRS transformation, $\mathbf{T}_{IRS}$, may be conveniently written as [2]

$$\mathbf{T}_S = \mathbf{T}S + \mathbf{S}\mathbf{M}\mathbf{T}_s\mathbf{M}_R^{-1}\mathbf{K}_R, \tag{4}$$

where

$$\mathbf{S} = \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ss}^{-1}\end{bmatrix}.$$

The reduced mass and stiffness matrices in the IRS method are then

$$\mathbf{M}_{IRS} = \mathbf{T}_{IRS}^{\mathrm{T}}\mathbf{M}\mathbf{T}_{IRS}, \qquad \mathbf{K}_{IRS} = \mathbf{T}_{IRS}^{\mathrm{T}}\mathbf{K}\mathbf{T}_{IRS}. \tag{5}$$

Although equation (4) is a convenient form for expressing the IRS transformation, in practice it is inefficient to compute the transformation in this way. The IRS method may be extended by using dynamic rather than static reduction [3]. The transformation is then exact at a non-zero frequency chosen by the analyst, rather than the zero frequency in the standard IRS method.

3. THE ITERATED IRS METHOD

The $(i+1)$th transformation in the iterated IRS algorithm may be obtained from the ith transformation as

$$\mathbf{T}_{i+1} = \begin{bmatrix}\mathbf{I} \\ \mathbf{t}_{i+1}\end{bmatrix}, \tag{6}$$

where

$$\mathbf{t}_{i+1} = \mathbf{t}_s + \mathbf{K}_{ss}^{-1}[\mathbf{M}_{sm} \quad \mathbf{M}_{ss}]\mathbf{T}_i \mathbf{M}_{Ri}^{-1}\mathbf{K}_{Ri}, \tag{7}$$

$\mathbf{t}_0 = \mathbf{t}_s = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}$ is the static transformation, and the reduced mass and stiffness matrix at the ith iteration are defined as

$$\mathbf{M}_{Ri} = \mathbf{T}_i^{\mathrm{T}}\mathbf{M}\mathbf{T}_i, \qquad \mathbf{K}_{Ri} = \mathbf{T}_i^{\mathrm{T}}\mathbf{K}\mathbf{T}_i. \tag{8}$$

If the iterative procedure converges, then on convergence the solution will not change from one iteration to the next. The converged solution may be obtained from equations (6)–(8) by ensuring $\mathbf{T}_i = \mathbf{T}_{i+1} = \mathbf{T}$. This solution is then given by

$$\mathbf{t} = \mathbf{t}_s + \mathbf{K}_{ss}^{-1}[\mathbf{M}_{sm} \quad \mathbf{M}_{ss}]\mathbf{T}\mathbf{M}_R^{-1}\mathbf{K}_R, \tag{9}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} \\ \mathbf{t} \end{bmatrix} \tag{10}$$

and

$$\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}, \qquad \mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}. \tag{11}$$

4. RELATIONSHIP WITH SEREP

The System Equivalent Reduction Expansion Process (SEREP) is a reduction transformation based on a subset of the modes of the full structure. The reduced model reproduces the selected modes at the chosen master degrees of freedom [6, 7]. The SEREP transformation is defined, when there are more master degrees of freedom than modes of interest, as

$$\mathbf{T} = \begin{bmatrix} \mathbf{\Phi}_m \\ \mathbf{\Phi}_s \end{bmatrix}[\mathbf{\Phi}_m^{\mathrm{T}}\mathbf{\Phi}_m]^{-1}\mathbf{\Phi}_m^{\mathrm{T}}, \tag{12}$$

where $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_s$ are the modes of interest at the master and slave degrees of freedom. Obviously, if the number of master degrees of freedom equals the number of modes of interest then equation (12) may be simplified to

$$\mathbf{T} = \begin{bmatrix} \mathbf{\Phi}_m \\ \mathbf{\Phi}_s \end{bmatrix}\mathbf{\Phi}_m^{-1}.$$

The aim of this section is to show that the converged solution given by equations (9)–(11) is identical to the SEREP transformation. It is not immediately apparent or obvious that a converged solution exists, or that the iterated IRS method even converges, but this is considered in detail in the next section. In the following the number of master degrees of freedom is assumed to equal the number of modes of interest. Although this is generally not the case in SEREP transformations, the number of modes approximated by the iterated IRS transformation is fixed by the number of master degrees of freedom.

Substituting the SEREP transformation into equations (9) and (10), and applying the transformation to $\mathbf{\Phi}_m$, gives

$$\mathbf{T\Phi}_m = \mathbf{T}_s\mathbf{\Phi}_m + \mathbf{SMTM}_R^{-1}\mathbf{K}_R\mathbf{\Phi}_m = \mathbf{T}_s\mathbf{\Phi}_m + \mathbf{SMT\Phi}_m\mathbf{\Lambda}_m$$

$$= \mathbf{T}_s\mathbf{\Phi}_m + \mathbf{SM}\begin{bmatrix}\mathbf{\Phi}_m\\ \mathbf{\Phi}_s\end{bmatrix}\mathbf{\Lambda}_m = \mathbf{T}_s\mathbf{\Phi}_m + \mathbf{SK}\begin{bmatrix}\mathbf{\Phi}_m\\ \mathbf{\Phi}_s\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{I}\\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\end{bmatrix}\mathbf{\Phi}_m + \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_m\\ \mathbf{\Phi}_s\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}_m\\ \mathbf{\Phi}_s\end{bmatrix}, \tag{13}$$

where $\mathbf{\Lambda}_m$ is a diagonal matrix of the eigenvalues corresponding to the modes $\mathbf{\Phi}_m$. The derivation in equation (13) has used the properties of the SEREP; namely, that the reduced model reproduces the modes of interest. This demonstrates that the SEREP transformation does indeed satisfy equation (9) and is a solution to the converged iterated IRS procedure.

5. CONVERGENCE OF THE ITERATED IRS METHOD

Convergence will be demonstrated by showing that $\mathbf{T}_i$ converges to $\mathbf{T}$. It is assumed that the SEREP transformation uses the lower modes of the full system and that the number of master co-ordinates equals the number of modes. Generally fewer modes are used in the SEREP transformation, but the number of degrees of freedom in the reduced model obtained by using the iterated IRS transformation equals the number of master co-ordinates. Define

$$\mathbf{r}_i = \mathbf{t}_i - \mathbf{t} \qquad \text{and} \qquad \mathbf{R}_i = \mathbf{T}_i - \mathbf{T} = \begin{bmatrix}\mathbf{0}\\ \mathbf{r}_i\end{bmatrix}, \tag{14}$$

so that it is necessary to show that $\mathbf{R}_i$ converges to zero. Now, from equations (7) and (9),

$$\mathbf{r}_{i+1} = \mathbf{K}_{ss}^{-1}[\mathbf{M}_{sm} \quad \mathbf{M}_{ss}][\mathbf{T}_i\mathbf{M}_{Ri}^{-1}\mathbf{K}_{Ri} - \mathbf{TM}_R^{-1}\mathbf{K}_R]. \tag{15}$$

To first order in $\mathbf{R}_i$, from equations (8) and (14), upon assuming that the mass and stiffness matrices are symmetrical,

$$\mathbf{T}_i\mathbf{M}_{Ri}^{-1}\mathbf{K}_{Ri} = [\mathbf{T} + \mathbf{R}_i][\mathbf{M}_R + \mathbf{T}^{\mathrm{T}}\mathbf{MR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{MT}]^{-1}[\mathbf{K}_R + \mathbf{T}^{\mathrm{T}}\mathbf{KR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{KT}]$$
$$= \mathbf{TM}_R^{-1}\mathbf{K}_R + \mathbf{R}_i\mathbf{M}_R^{-1}\mathbf{K}_R$$
$$+\mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{KR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{KT}] - \mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{MR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{MT}]\mathbf{M}_R^{-1}\mathbf{K}_R. \tag{16}$$

Thus combining equations (14), (15) and (16) yields

$$\mathbf{R}_{i+1} = \mathbf{SM}[\mathbf{R}_i\mathbf{M}_R^{-1}\mathbf{K}_R + \mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{KR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{KT}] - \mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{MR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{MT}]\mathbf{M}_R^{-1}\mathbf{K}_R]. \tag{17}$$

Let λ_j be the jth eigenvalue of the full model with corresponding eigenvector ϕ_j. Since $\mathbf{T}$ is based on SEREP, if ϕ_{mj} and ϕ_{sj} are this mode restricted to the master and slave co-ordinates, respectively, then

$$\phi_{sj} = \mathbf{t}\phi_{mj}, \qquad \phi_j = \mathbf{T}\phi_{mj}, \qquad \lambda_j\mathbf{M}_R\phi_{mj} = \mathbf{K}_R\phi_{mj}. \tag{18}$$

Thus the transformation (17) may be applied to the jth mode, as follows:

$$\begin{aligned}\mathbf{R}_{i+1}\phi_{mj} &= \mathbf{SM}[\mathbf{R}_i\mathbf{M}_R^{-1}\mathbf{K}_R + \mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{KR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{KT}] - \mathbf{TM}_R^{-1}[\mathbf{T}^{\mathrm{T}}\mathbf{MR}_i + \mathbf{R}_i^{\mathrm{T}}\mathbf{MT}]\lambda_j]\phi_{mj}\\ &= \mathbf{SM}[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\mathbf{T}^{\mathrm{T}}[\mathbf{K}-\lambda_j\mathbf{M}]\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\mathbf{R}_i^{\mathrm{T}}[\mathbf{K}-\lambda_j\mathbf{M}]\mathbf{T}\phi_{mj}]\\ &= \mathbf{SM}[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\mathbf{T}^{\mathrm{T}}[\mathbf{K}-\lambda_j\mathbf{M}]\mathbf{R}_i\phi_{mj}].\end{aligned} \tag{19}$$

In equation (19), the identity $[\mathbf{K}-\lambda_j\mathbf{M}]\mathbf{T}\phi_{mj}=\mathbf{0}$, from the properties of the SEREP transformation, is used. $\mathbf{R}_i\phi_{mj}$ is now written as a combination of the full eigenvectors of the structure. Note that this sum cannot be restricted just to the sum over those eigenvectors of the reduced model, although the index j in equation (19) does range only over the reduced model modes. Thus, suppose that

$$\mathbf{R}_i\phi_{mj} = \sum_k a_{ijk}\phi_k. \tag{20}$$

Substituting this into equation (19) gives

$$\begin{aligned}\mathbf{R}_{i+1}\phi_{mj} &= \mathbf{SM}\left[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\mathbf{T}^{\mathrm{T}}[\mathbf{K}-\lambda_j\mathbf{M}]\sum_k a_{ijk}\phi_k\right]\\ &= \mathbf{SM}\left[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\sum_k a_{ijk}(\lambda_k-\lambda_j)\mathbf{T}^{\mathrm{T}}\mathbf{M}\phi_k\right].\end{aligned} \tag{21}$$

In equation (21), the last summation has been written in that form because the definition of the SEREP transformation and mass orthogonality may be used to simplify the equation. Let $\mathfrak{I}_m$ denote the set of modes in the reduced model. Then

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\phi_k = \mathbf{0}, \qquad \text{for } k\notin\mathfrak{I}_m. \tag{22}$$

Thus,

$$\begin{aligned}\mathbf{R}_{i+1}\phi_{mj} &= \mathbf{SM}\left[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\sum_{k\in\mathfrak{I}_m} a_{ijk}(\lambda_k-\lambda_j)\mathbf{T}^{\mathrm{T}}\mathbf{M}\phi_k\right]\\ &= \mathbf{SM}\left[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{TM}_R^{-1}\sum_{k\in\mathfrak{I}_m} a_{ijk}(\lambda_k-\lambda_j)\mathbf{T}^{\mathrm{T}}\mathbf{MT}\phi_{mk}\right]\\ &= \mathbf{SM}\left[\lambda_j\mathbf{R}_i\phi_{mj} + \mathbf{T}\sum_{k\in\mathfrak{I}_m} a_{ijk}(\lambda_k-\lambda_j)\phi_{mk}\right]\\ &= \mathbf{SM}\left[\sum_k a_{ijk}\lambda_j\phi_k + \sum_{k\in\mathfrak{I}_m} a_{ijk}(\lambda_k-\lambda_j)\phi_k\right]\\ &= \mathbf{SM}\left[\sum_{k\in\mathfrak{I}_m} a_{ijk}\lambda_k\phi_k + \sum_{k\notin\mathfrak{I}_m} a_{ijk}\lambda_j\phi_k\right].\end{aligned} \tag{23}$$

Use of the properties of the mode shape and the definition of **S** yields

$$\mathbf{R}_{i+1}\phi_{mj} = \mathbf{SK}\left[\sum_{k\in\Im_m} a_{ijk}\phi_k + \sum_{k\notin\Im_m} a_{ijk}\frac{\lambda_j}{\lambda_k}\phi_k\right]$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{t}_s & \mathbf{I} \end{bmatrix}\left[\sum_{k\in\Im_m} a_{ijk}\phi_k + \sum_{k\notin\Im_m} a_{ijk}\frac{\lambda_j}{\lambda_k}\phi_k\right]$$

$$= \sum_{k\in\Im_m} a_{ijk}\begin{Bmatrix} \mathbf{0} \\ \phi_{sk} - \mathbf{t}_s\phi_{mk} \end{Bmatrix} + \sum_{k\notin\Im_m} a_{ijk}\frac{\lambda_j}{\lambda_k}\begin{Bmatrix} \mathbf{0} \\ \phi_{sk} - \mathbf{t}_s\phi_{mk} \end{Bmatrix}. \tag{24}$$

The first term in equation (24) will be small because the expanded modes from static reduction should be close to the modes at the slave degrees of freedom and so $\{\phi_{sk} - \mathbf{t}_s\phi_{mk}\}$ will be small. In the second term this vector will not necessarily be small, since the static reduction will not accurately reproduce the higher modes at the slave degrees of freedom. The second term becomes smaller because the ratio λ_j/λ_k will be less than unity (remember that the index k runs over the modes not included in the SEREP reduced model). Thus $\mathbf{R}_{i+1}\phi_{mj}$ will be significantly smaller than $\mathbf{R}_i\phi_{mj}$.

Equation (24) shows that the best degrees of freedom to use are the same as in static reduction. Furthermore, the lower modes will converge more quickly than the higher modes, since in equation (24) for the lower modes the ratio λ_j/λ_k will be significantly smaller.

6. CHOOSING THE MASTER CO-ORDINATES

The question arises as to how the master co-ordinates are to be selected. Here the underlying assumption in Guyan reduction must be borne in mind: that at slave co-ordinates the inertia forces are negligible compared to the elastic forces. Thus the slaves should be chosen where the inertia is low and the stiffness is high so that the mass is well connected to the structure. Conversely, the master co-ordinates are chosen where the inertia is high and the stiffness is low. This process can be automated [8] by examining the ratio of the diagonal terms in the stiffness and mass matrices, k_{ii}/m_{ii}, for the ith co-ordinate. If k_{ii}/m_{ii} is small then there are significant inertia effects associated with this co-ordinate and thus it should be retained as a master; if k_{ii}/m_{ii} is large, then the ith co-ordinate should be chosen as a slave and removed.

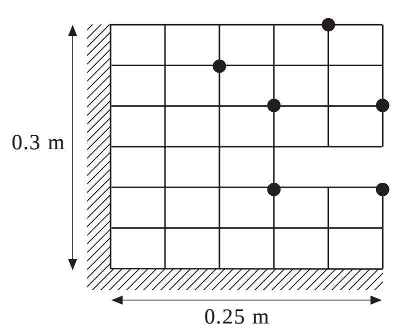

Figure 1. The plate example. The dots represent the optimum set of master co-ordinates (all masters are translations out of the plane of the plate).

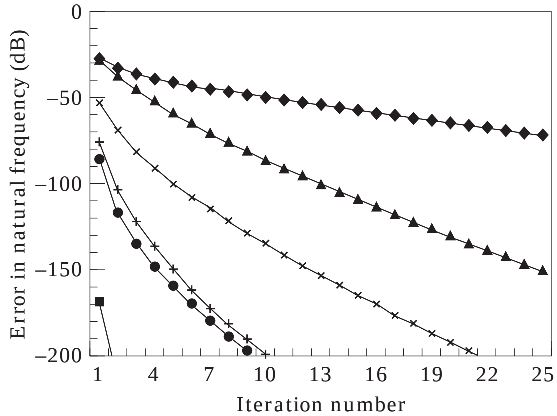

Figure 2. The convergence of the first six natural frequencies for the optimum choice of master co-ordinates. Frequency number: –■–, 1; –●–, 2; –|–, 3; –×–, 4; –▲–, 5; –◆–, 6.

The slave co-ordinates are not chosen according to the above rule *en bloc*, but rather chosen and removed one at a time. There are two advantages to this procedure. First, at each stage the effect of each co-ordinate removed is redistributed to all the remaining co-ordinates so that the next reduction will remove the co-ordinates with the highest k_{ii}/m_{ii} ratio in the reduced mass and stiffness matrices. Second, there is a very simple algorithm for performing this sequential process of co-ordinate selection and removal.

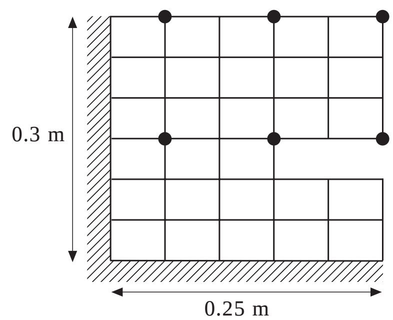

Figure 3. As Figure 1, except that the dots represent the arbitrary set of master co-ordinates.

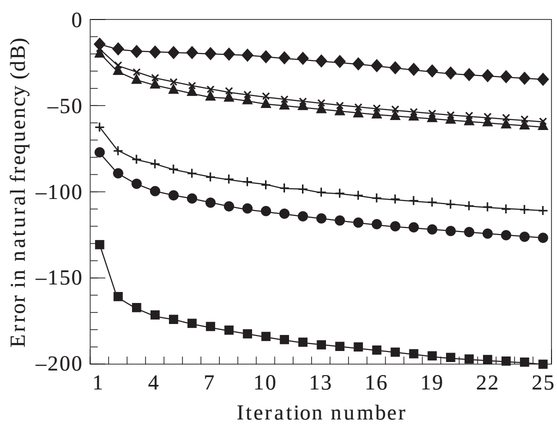

Figure 4. As Figure 2, except that it is for a poor choice of master co-ordinates.

TABLE 1

Natural frequencies (in Hz) for the plate example; the natural frequencies of the reduced model are obtained by using static reduction

Mode number	Exact	Reduced model; good choice of masters	Reduced model; poor choice of masters
1	4·86	4·87	4·91
2	16·67	17·23	17·04
3	22·48	23·44	24·63
4	33·19	35·84	45·47
5	41·79	52·32	81·48
6	56·36	70·25	94·76

7. A NUMERICAL EXAMPLE

We now illustrate the application of the iterative IRS method to a discrete model of a continuous structure. The convergence of the method has already been demonstrated [3], and this example is intended to demonstrate the convergence properties of the method. In particular, the effect of the choice of master degrees of freedom is shown and the relative rates of convergence of the different modes is highlighted. The structure considered is a 1 mm thick, steel plate with a slot on one side and clamped along two other sides, as shown in Figure 1. The full model has 90 degrees of freedom and is reduced to the six degrees of freedom shown in Figure 1. This choice of co-ordinates represents the best selection of co-ordinates for Guyan reduction on the basis of the relative importance of the diagonal stiffness and inertia terms [8, 9]. In Table 1 are given the first six natural frequencies of the structure, and also the natural frequencies of the reduced model obtained by using static reduction. In Figure 2 is shown the convergence of the first six natural frequencies of the reduced model when using the iterated IRS technique. Convergence to the natural frequencies of the full model is clear. As expected, the lower modes converge more quickly than the higher modes and in practice one would retain approximately twice as many masters as the number of required eigenvalues [10]. In Figure 3 is shown the location of six arbitrary master co-ordinate locations, and in Figure 4 is shown the resulting convergence of the eigenvalues. Clearly, this set of master co-ordinates produces a slower convergence of the eigenvalues. The natural frequencies based on static reduction are shown in Table 1, and highlights the requirement that these co-ordinates should be chosen to produce a good reduced model by using static reduction.

8. CONCLUSIONS

It has been shown that the iterated IRS will converge to the SEREP transformation, which reproduces the lower modes of the full system. The convergence analysis and the numerical example both show that convergence is faster if master degrees of freedom are chosen as the optimum for static reduction. Furthermore, the lower modes will converge more quickly than the higher modes.

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