Advanced nonlinear dynamic modelling of bi-stable composite plates

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\textbf{A B S T R A C T}

This paper proposes a novel analytical model to study the nonlinear dynamics of cross-ply bi-stable composite plates. Based on Hamilton’s principle, in conjunction with the Rayleigh-Ritz method, an advanced analytical model with only 17 unknown terms is developed to predict the entire nonlinear dynamic response of bi-stable composite plates, which are excited by an electrodynamic shaker. The coupling between the bi-stable plate and the shaker is considered in the development of the analytical model. This work, for the first time, simulates the full dynamics of bi-stable plates using an analytical model, including the prediction of the nonlinear characteristics of single well vibration and cross well vibration. Numerical results on three vibrational patterns of two standard cross-ply composite plates are obtained to study the nonlinear dynamics of bi-stable plates. The prediction accuracy on the dynamic characteristics of different vibrational patterns of bi-stable plates are verified by both finite element analysis (FEA) and experimental results. Large amplitude cross-well vibrations due to the transitions between different stable states of bi-stable plates are also characterized accurately. Applying this 17-term analytical model for the dynamic analysis of bi-stable plates is straightforward, as the mass and stiffness properties are obtained directly from the geometry and material properties. Only the damping coefficients for different plates need to be determined from experiments. Furthermore, this proposed 17-term analytical model has much higher computational efficiency than FEA.

1. Introduction

Thin unsymmetric composite plates which exhibit multiple stable static configurations have attracted considerable attention for over three decades [1–4]. A bi-stable fiber reinforced composite plate possesses two stable equilibria, and the plate is able to maintain either of the equilibria without external forces. The bi-stability of composite plates enables large deflections from one equilibrium state to another with small energy input. The static characteristics of unsymmetric bi-stable fiber reinforced plates have been well quantified, after a comprehensive research effort over the last three decades. Bi-stable composite plates have shown extensive potential applications for morphing aircraft [5–10] and energy harvesting devices [11–13].

Morphing structures change shape to continually optimize operating conditions, according to the environment or loads. In the morphing concepts that take advantage of bi-stable plates, the equilibrium states are matched to shape requirements. Several aero-structures applications with bi-stable morphing airfoils have been proposed and have been experimentally and numerically investigated [5–10]. More recently, research works shed light on the dynamic excitation that can trigger the bi-stability of composite plates from one equilibrium state to another with small energy input, for example [14–17]. Recently, the dramatic increase of wireless sensors and electronics, has meant that energy harvesting from ambient vibration has become an active research topic [18,19]. Nonlinear piezoelectric harvesters have been proposed for broadband energy harvesting. Energy harvesters that employ bi-stable plates as the supporting structure have been proposed. The natural characteristics of bi-stable plates, including the nonlinear bending stiffness and the snap-through phenomenon, are used to increase the power generated and increase the effective frequency range of operation. In particular, the snap-through between equilibrium states gives large velocities that increases the output power [19]. For morphing applications, bi-stable plates are inevitably exposed to high levels of dynamic excitation, and early failure may be induced. Moreover, undesired snap-through may be triggered by the dynamic excitation. For energy harvesting applications, complex vibration patterns including single-well oscillations, chaos and limit cycles are observed in experiments [20]. In the last decade the investigation of nonlinear...
The bi-stable plates are assumed to be thin. Thus, the Kirchhoff hypothesis is employed and the layers are assumed to be in the state of plane stress [28]. Hence

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} + \begin{bmatrix}
k_1 \\
k_2 \\
k_3
\end{bmatrix} \varepsilon
\]

(2)

where

\[
k = \begin{bmatrix}
k_1 \\
k_2 \\
k_3
\end{bmatrix}
= \begin{bmatrix}
-\frac{\nu^2}{2} & \frac{1}{2}
\frac{\nu^2}{2} & -\frac{1}{2}
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

(3)

represents the plate curvatures. The deflections of bi-stable plates are relative large compared to the plate thicknesses, and hence the geometrical nonlinearity has to be considered in the geometric equation [29]. The mid-plane strains, \(\varepsilon^0\), are then defined in the spirit of von Karman, and the in-plane strains of a thin bi-stable plate take the form

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial \nu}{\partial x} \\
\frac{\partial^2 w}{\partial y^2} + \frac{\partial \nu}{\partial y} \\
\frac{\partial w}{\partial x} \frac{\partial \nu}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \nu}{\partial x}
\end{bmatrix}
\]

(4)

where \(u^0, v^0\) and \(w\) denote the in-plane and out-of-plane displacements. The total potential energy of the plate under a thermal-induced load is given by

\[
\Pi = \int_{L_2} \int_{L_1} \int_{L_2} \frac{1}{2} \left(\varepsilon^0 \cdot K \varepsilon^0 \right) \left( \begin{array}{c}
A & B & 0 \\
B & D & 0 \\
0 & 0 & N^f
\end{array} \right) \left( \begin{array}{c}
\varepsilon^0 \\
\nu^0 \\
\nu^f
\end{array} \right) dxdy
\]

(5)

where \(L_1\) and \(L_2\) denote plate length and width, respectively, and the explicit time dependence has been omitted. \(A, B\) and \(D\) are the stretching stiffness matrix, the stretching-bending coupling matrix and the bending stiffness matrix, respectively. \(N\) and \(M\) are the resultant force and moment due to the thermal stress. The potential energy depends on the in-plane strains \(\varepsilon^0\), and the curvatures \(\kappa\), which are determined by the assumed displacement functions, \(u^0, v^0\) and \(w\).

The mid-plane strain, \(\varepsilon^0\), and the bending curvature, \(\kappa\), for an unsymmetric plate are coupled, since the stretching-bending coupling matrix, \(B\), is non-zero. If a displacement model for the out-of-plane displacement function, \(w\), is assumed, then the mid-plane strains, \(\varepsilon^0\) and \(\varepsilon^0_w\), should have terms which are dependent on \(w\), due to the stretching-bending coupling. However, previous studies often applied a generalized modelling approach for unsymmetric plates, where \(\varepsilon^0, \varepsilon^0_w\) and \(w\) are defined by independent series [30,31]. However, the influence of stretching-bending coupling on the assumed form of the mid-plane strains, \(\varepsilon^0\) and \(\varepsilon^0_w\), were not explicitly considered. As a result, more terms have to be used in the series expansion of mid-plane strains to achieve better prediction accuracy.

In this study, an alternative procedure is applied to model the mid-plane strains \(\varepsilon^0\) for cross-ply bi-stable plates. According to classical lamination theory, the in-plane stress resultants for a thermally-loaded unsymmetric plate is expressed as

\[
N = A\varepsilon^0 + B\kappa - N^f
\]

(6)

Eq. (6) can be rearranged to give

\[
\varepsilon^0 = A^tN + A^tN^f + A^tBk = \varepsilon^m + A^tN^f + A^tBk
\]

(7)

where \(A^tN\) can be essentially considered as a membrane strain field \(\varepsilon^m\). Substituting Eq. (7) into Eq. (5), the total strain energy becomes
The mid plane displacement functions, \(u^0(t)\) and \(v^0(t)\), are then derived by integrating with \(x\) and \(y\), as
\[
\begin{align*}
\frac{du^0}{dt}(t) &= \int \varepsilon_{xx}^0(t)^2 - \frac{1}{2} B_1^0 k_1 + B_2^0 k_2 + A^{-1} N_1^T \ dy \\
\frac{dv^0}{dt}(t) &= \int \varepsilon_{yy}^0(t)^2 - \frac{1}{2} B_1^0 k_1 + B_2^0 k_2 + A^{-1} N_1^T \ dx
\end{align*}
\] (14)

The shear component of mid plane and stretching strains \(\gamma_{xy}^0(t)\) and \(\gamma_{yy}^0(t)\) are derived by substituting the series forms of \(u^0(t)\), \(v^0(t)\) and \(w(t)\) into Eq. (2). The polynomial expressions for \(u^0(t)\), \(v^0(t)\) and \(w^0(t)\) are given in the Appendix A. Note, the constants in \(u^0(t)\) and \(v^0(t)\) due to integration of strains are set to 0. Substituting the expressions for \(\varepsilon_{xx}^0(t)\), \(\varepsilon_{yy}^0(t)\) and \(\gamma_{xy}^0(t)\) into Eq. (8), the dynamic potential energy \(\Pi(t)\) is derived. Since the mid plane shear strain \(\gamma_{xy}^0(t)\) is derived explicitly from the displacement functions, the compatibility condition for the mid-plane strains is satisfied automatically.

The bi-stable composite plates in this study are mounted and excited at the central point. The kinetic energy is defined as
\[
T(t) = \frac{1}{2} \rho h \int d \frac{w(t)}{0}^L \left( \frac{m_0}{w(t) + w(t)^2} \right)^2 \ dy \\
\] (15)

where \(\rho\) denotes the mass density, \(w_0(t)\) denotes the excitation displacement at the central point, and \(h\) is the thickness of the plate. In Eq. (15), the energy contributions given by the transverse inertia terms are neglected.

In the present study, four constant and equal concentrated transverse forces are applied at the four corners of the plate. In this case, the work done by the applied forces, \(W_p\), is
\[
\begin{align*}
W_p(t) &= 4F \left( \frac{L}{2} \right)^2 + b(t) \left( \frac{L}{2} \right)^2 + a_1 \left( \frac{L}{2} \right)^4 + b_1 \left( \frac{L}{2} \right)^4 \\
&+ a_2 \left( \frac{L}{2} \right)^6 + b_2 \left( \frac{L}{2} \right)^6 + c(t) \left( \frac{L}{2} \right)^8 + d(t) \left( \frac{L}{2} \right)^8
\end{align*}
\] (16)

3. Excitation controlled by displacement

In the present study, a sinusoidal excitation is applied at the centre of the bi-stable plate in the transverse direction. If the bi-stable plate is connected to a structure with greater mass and stiffness, the influence from the vibration of the supporting structure can be ignored. In this case, a prescribed excitation displacement \(w_0(t)\) is introduced to represent the motion of the supporting structure in the dynamic model. Since there are 17 degrees of freedom in the proposed dynamic model, the generalized displacement is expressed as
\[
X_i(t) = [a, b, a_1, a_2, b_1, b_2, e, c, d, c_1, d_1, c_2, d_2, c_3, d_3, c_4, d_4]^T
\] (17)

According to the Hamilton principle, the variation of the Lagrangian energy function is expressed as
\[
\Pi(t) = \int \left\{ \frac{1}{2} \varepsilon_{xx}^0(t)^2 - \frac{1}{2} B_1^0 k_1 + B_2^0 k_2 + A^{-1} N_1^T \right\} \ dy \\
\] (8)
function, given by \((T - \Pi - W_v)\), is equal to zero. The 17 equations of motion are thus derived, and expressed in the following matrix form

\[
\mathbf{M}_1 \ddot{\mathbf{X}}_1 + \mathbf{D}(\mathbf{X}_1) + \mathbf{K}(\mathbf{X}_1) = \mathbf{F}_1 \tag{18}
\]

where \(\mathbf{M}_1\) is the mass matrix, \(\mathbf{D}(\mathbf{X}_1)\) is the damping force, \(\mathbf{K}(\mathbf{X}_1)\) is the nonlinear stiffness force, and the overdot denotes differentiation with respect to time. \(\mathbf{F}_1\) denotes the excitation force and also can be expressed by \(\mathbf{F}_1 = \mathbf{G} \dot{w}_0\), which is the base excitation force, which depends on the plate inertia and the acceleration \(\ddot{w}\). The vector \(\mathbf{G}\) is given in the Appendix A.

With the definition of the generalized co-ordinates in Eq. (17), and the definition of the displacement models for \(u(t), v(t)\) and \(w(t)\) given by Eqs. (9) and (14), the mass matrix \(\mathbf{M}_1\) is easily derived. Similarly, the stiffness matrix \(\mathbf{K}_1\) is derived by writing the strain energy of the plate, \(\Pi\), in terms of \(\mathbf{X}_1\), and differentiating in the usual way. The expressions of the matrices \(\mathbf{M}_1\) and \(\mathbf{K}_1\) are given in Appendix A. Since the displacement \(w(t)\) is a linear function of the degrees of freedom, the mass matrix \(\mathbf{M}_1\) is a constant matrix.

Damping is very difficult to model precisely, and here the assumption of Rayleigh proportional damping \([24]\) is employed and thus:

\[
\mathbf{D}(\mathbf{X}_1, \dot{\mathbf{X}}_1) = \alpha \mathbf{M}_1 \mathbf{X}_1 + \beta \mathbf{M}_1 \dot{\mathbf{X}}_1 \tag{19}
\]

where \(\alpha\) is the mass proportional damping coefficient. As the vibration of bi-stable plates often occurs at low frequencies, the contribution of stiffness damping is ignored. Thus, only the mass damping effect is considered in this study, and the damping force is a function of velocity alone.

4. Excitation controlled by force from a shaker

In previous studies on the dynamics of bi-stable plates, electrodynamic shakers were often used to excite the plates. Arrieta et al. used a transducer between the plate and the shaker to measure the excitation force, and a linear feedback proportional controller was implemented to eliminate the dynamic coupling between the plate and the shaker \([23,34]\). As the dynamics of bi-stable plates is nonlinear and complex, it is unlikely that the coupling between the plate and the shaker can be eliminated, completely. In this study, the coupling between the plate and the shaker is explicitly considered using the following analytical model. An electromagnetic shaker consists of a moving part, the armature assembly, a linear spring support for the armature and a dashpot, as illustrated in Fig. 2.

The single degree of freedom dynamic model of the shaker is given by

\[
m \ddot{w}_0 + C \dot{w}_0 + k w_0 = f(t) \tag{20}
\]

where \(m\) is the mass of the moving armature, \(C\) is the damping coefficient of the dashpot, \(k\) is the stiffness of the spring system, and \(f(t)\) denotes the sinusoidal electromagnetic force of the shaker. Due to the coupling between the shaker and the plate, the oscillation amplitude of the centre of the plate, \(w_0\), is unknown. Therefore, the system is a combination of the bi-stable plate and the shaker. There are now 18 degrees of freedom as \(w_0\) is now also included. Thus,

\[
\mathbf{X}_1 = \begin{bmatrix} w_0 \\ \mathbf{X}_1 \end{bmatrix} \tag{21}
\]

The dynamic equations are derived similarly as the process of deriving Eq. (18),

\[
\mathbf{M}_1 \ddot{\mathbf{X}}_1 + \mathbf{C}(\mathbf{X}_1) + \mathbf{K}(\mathbf{X}_1) = \mathbf{F}_2 \tag{22}
\]

Each term in Eq. (22) is expressed in matrix form as

\[
\mathbf{C} = \begin{bmatrix} C & 0 \\ 0 & \alpha \mathbf{M}_1 \end{bmatrix} \tag{23}
\]

\[
\mathbf{K}_1(\mathbf{X}_1) = \begin{bmatrix} k \mathbf{X}_1(1) \\ \mathbf{K}_i(\mathbf{X}_i) \end{bmatrix} \tag{24}
\]

\[
\mathbf{F}_2(t) = [f(t) \ 0 \ldots \ 0]^T \tag{25}
\]

The complete form of \(\mathbf{M}_2\) is presented in the Appendix A.

5. The effect of inertial mass

Inertial masses are often attached to the corners of bi-stable plates to increase the dynamic response. The contribution of these masses (assumed to be equal at each corner) to the kinetic energy is

\[
\mathcal{T}_{\text{inertial}} = \frac{1}{2} m_{\text{inertial}} \left[ \sum_{i=1}^{4} \dot{\psi}_i^2 + \sum_{i=1}^{4} \dot{\phi}_i^2 + \sum_{i=1}^{4} \dot{\theta}_i^2 \right] \tag{26}
\]

where the velocities of each mass (i.e. each corner) are

\[
\dot{\psi}_1, ..., \dot{\psi}_4(t) = \dot{\psi}\left( \pm \frac{\sqrt{2}}{2} l_1, \pm \frac{\sqrt{2}}{2} l_2, 1 + \ddot{w}_0(t) \right)
\]

\[
\dot{u}_1, ..., \dot{u}_4(t) = \dot{u}\left( \pm \frac{\sqrt{2}}{2} l_1, \pm \frac{\sqrt{2}}{2} l_2, 1 \right)
\]

\[
\dot{v}_1, ..., \dot{v}_4(t) = \dot{v}\left( \pm \frac{\sqrt{2}}{2} l_1, \pm \frac{\sqrt{2}}{2} l_2, 1 \right) \tag{27}
\]

and \(m_{\text{inertial}}\) represents the mass added to each corner of the plate. The contribution to the system mass matrix is easily derived. In Eq. (18), the contribution of inertial mass is added to the mass matrix by replacing \(\mathbf{M}_1\) with \(\mathbf{M}_1 + m_{\text{inertial}}\). Similarly, the mass matrix \(\mathbf{M}_2\) should be replaced by \(\mathbf{M}_2 + m_{\text{inertial}}\) in Eq. (22). The matrices \(m_{\text{inertial1}}\) and \(m_{\text{inertial2}}\) are given in the Appendix A. For the excitation force in Eq. (18), the contribution of the inertial mass \(\mathbf{F}(\mathbf{X})_{\text{inertial}}\) is given in the Appendix A, and \(\mathbf{F}(\mathbf{X})\) in Eq. (8) is replaced by \(\mathbf{F}(\mathbf{X}) + \mathbf{F}(\mathbf{X})_{\text{inertial}}\).

6. Experimental setup

In the present study, an experiment is established to test the dynamic response of bi-stable composite plates. Fig. 3(a) illustrates the schematic diagram of this experimental system, in which each experimental device is connected with respect to the signal (data) flow. Fig. 3(b) shows the practical experimental set up, in which the bi-stable plate is mounted at its centre and excited by a shaker. An accelerometer is attached at the centre of the bi-stable plate to measure the excitation acceleration. A laser vibrometer is used to measure the full dynamic response of the bi-stable plate, and particularly the motion at the plate corners where the 11.54 g inertial masses are attached.
7. Finite element analysis

In this study, finite element analysis (FEA) for the bi-stable plates was performed using the commercial tool ABAQUS to validate the proposed analytical model. The S4R shell element (4-node general-purpose shell, reduced integration with hourglass control, finite membrane strains) was chosen to model the bi-stable plates. A mesh density of \( 4040 \times \) was chosen in the finite element model (FEM) to achieve the required accuracy and efficiency.

A static analysis with two different conditions predicted the stable configurations for the bi-stable plates. In the first "Static" step, a stable configuration is determined under the conditions that the plate cools down from the curing temperature to room temperature, and a concentrated force along transverse direction is applied at each corner. In the second "Static" step, another stable configuration is obtained for the bi-stable plate, where the transverse concentrated forces are removed, the temperature is kept constant and the second stable configuration is obtained.

After the static analysis, the nonlinear dynamic analysis is performed for the bi-stable plates using the "Dynamic implicit" step in ABAQUS. The plate is mounted at the centre and excited at the centre in the transverse direction. For the case of prescribed excitation displacement, a sinusoidal mass is applied to the centre of the plate, and a sinusoidal force is applied to the plate centre in the transverse direction, as illustrated in Fig. 4.

8. Results and discussion

The bi-stable plates are manufactured using the unidirectional carbon fiber-epoxy matrix prepreg, CCF300/5428. The material properties of the CCF300/5428 prepreg are given in Table 1.

8.1. Static analysis

The static equilibrium shapes of the bi-stable plates need to be determined as the initial states for dynamic analysis. The static equilibrium configurations of the bi-stable plates will be obtained by solving the nonlinear coupled equations

\[
K(X) = 0
\]  

(28)

The "FindRoot" function with initial values of \( X \) is employed in Mathematica® to solve Eq. (28). For the bi-stable plates, there are three groups of solutions, and each group of solutions represents an equilibrium configuration. The actual equilibrium configuration depends on the initial value of the unknowns \( X \). The stability of the equilibrium solutions need to be checked by the Jacobian matrix, which is positive definite for stable configurations:

\[
J(\delta k(X)) = \delta(X)
\]  

(29)

The static stable configurations of the square composite plates \( 100 \text{ mm} \times 100 \text{ mm}, [0^\circ/90^\circ]_2 \) and \( 100 \text{ mm} \times 100 \text{ mm}, [0^\circ/90^\circ]_3 \) are analyzed in this section. Each plate is mounted at its centre, and concentrated forces are applied at each corner of the bi-stable plates. The plates cool down from the curing temperature of 180 °C to the room temperature of 20 °C. The stable configurations for the bi-stable plates are predicted by both the analytical model and the FEA. The two stable cylindrical mode shapes of each square bi-stable plate are identical in principle, and the only difference lies in the different orientation of the cylindrical deformation which are perpendicular to each other. Figs. 5 and 6 present the predicted stable states of the two bi-stable plates \( [0^\circ/90^\circ]_2 \) and \( [0^\circ/90^\circ]_3 \), respectively, and show that the stable configurations predicted by the two methods coincide quite well. The numerical results for the out-of-plane deflection at the corners of the bi-stable plates given by the analytical model and the FEA are compared in Table 2. The small errors between the predicted results from the two methods, as shown in Table 2, further validate the proposed analytical model proposed in this work.

8.2. Free vibration

To study the free vibration of the bi-stable plates, four identical concentrated forces are initially applied at each corner of the plates, which are mounted at their centres. The applied forces are the suddenly removed. For illustration purposes, the numerical results are only determined and presented for the \( 100 \text{ mm} \times 100 \text{ mm}, [0^\circ/90^\circ]_2 \) bi-stable plate. Four inertial masses of 11.54 g are attached to each corner of the
bi-stable plates to increase the amplitude of the dynamic response. The static stable configurations of the [0/90]T bi-stable plate predicted by the analytical model are shown in Fig. 6, and are used as the initial deformation for the dynamic analysis. The dynamic response of the bi-stable plates requires the solution of Eq. (18). The excitation amplitude at the centre of the plate is zero in the free vibration analysis, i.e. \( w_0(t) = 0 \). In the FEA, the concentrated forces are applied in the "Dynamic implicit" step, and are removed in the "Dynamic implicit" step. Damping is not considered in either analysis, i.e. \( \alpha = 0 \).

The free vibration of the [0/90]T bi-stable plate subjected to initial concentrated forces of 2 N, 0.5 N and 0 N are predicted. Fig. 7(a) and (b) present the oscillation curves at the plate corners determined by the analytical model and the FEA, respectively. Fig. 7 shows that the analytical results and the FEA results are very close to each other. When the initial deformation (applied force = 0.5 N) is small, the plate oscillation is nearly sinusoidal. However, with larger initial deformation (applied force = 2 N), the oscillation for the bi-stable plate exhibits slightly non-sinusoidal characteristics.

The Fast Fourier Transform (FFT) of the predicted vibration presented in Fig. 7 is shown in Fig. 8. For small amplitude excitation (0.5 N) there is only one peak, which indicates that the vibration is purely sinusoidal. The first dominant frequency for the analytical model is 44 Hz, which is slightly higher than that given by the FEA (42 Hz). With larger amplitude excitation (2 N), the FFTs have two peaks. The second dominant frequency is twice that of the first dominant frequency, and this indicates the second harmonic vibration. The peak frequencies predicted by the analytical model (40 Hz and 80 Hz) are slightly higher than those predicted by FEA (38 Hz and 76 Hz). Moreover, the FFT results in Fig. 8 also illustrate that the first dominant frequency decreases by 4 Hz when the oscillation amplitude increases (the applied load increases from 0.5 N to 2 N). This indicates that the stiffness of the bi-stable plate decreases with respect to increasing oscillation amplitude (i.e. softening), due to the natural nonlinearity of the bi-stable plate. Although the analytical model over predicts the dominant frequencies by about 2 Hz, it accurately predicts the nonlinearity of the bi-stable plate. If damping is considered in both the analytical model and the FEM with \( \alpha = 100 \), the predicted free vibration of the 100 mm \( \times \) 100 mm, [0/90]T plate shown in Fig. 9 is obtained. The decay rates of the vibrations predicted by the two models are nearly identical.

### 8.3. Single well vibration

The forced vibration of the bi-stable plates within a single energy well is investigated by experiment, analytical model and FEA. The experimental setup for the vibration testing of bi-stable plates is illustrated in Fig. 3. The excitation acceleration at the plate centre was monitored throughout the experiments. For the low amplitude single well vibration, the excitation acceleration at the plate centre is sinusoidal. The excitation amplitude is adjusted manually from zero to the target level at each frequency. The oscillation amplitudes at the plate corner at different frequencies were recorded by a laser vibrometer, with a fixed level of excitation acceleration for each measurement. The measured vibration amplitudes at the plate corners for the two bi-stable plates ([0/90]T and [0/90]T) are presented in Fig. 10.

Eq. (18) is solved to predict the small amplitude single well vibration of bi-stable plates. Diaconu et al. [24] suggested a measured mass damping coefficient of \( \alpha = 83 \), although \( \alpha \) depends on the plate dimensions, material properties, plate lay-up, etc. Preliminary studies indicate that changing \( \alpha \) at these excitation magnitudes has a negligible effect on the stable dynamics of single well vibration if the excitation time is sufficiently long. This benchmark study focuses on the validation of the analytical model against the FEA rather than precisely matching the experimental results. Therefore, a damping coefficient of \( \alpha = 100 \) is employed. The vibration amplitudes at the plate corners corresponding to different frequencies are predicted using the proposed analytical model and are also shown in Fig. 10.

The single well vibrations at different frequencies are also simulated by FEA. A sinusoidal displacement is applied at the centre of the bi-stable plates in the transverse direction. Compared with the analytical model, the time domain analysis in FEA using the “dynamic implicit” step is very time consuming. The oscillation amplitudes at the plate corners for different excitation frequencies are numerically determined by FEA and are compared with the results predicted by the analytical model, as shown in Fig. 10.

Fig. 10 shows that both the analytical model and the FEA predict only one response peak within the frequency range, whereas extra peaks were observed in the experimental results for each bi-stable plate. To study the origin of the extra resonant peaks in the experimental results, a modal analysis of the bi-stable plates mounted at the centre was performed using FEA. The mode shapes of a square [0/90]T bi-stable plate predicted by FEA are presented in Fig. 11. For the first and second modes, the bi-stable plate rotates around its centre. For the third mode, the plate exhibits a symmetric bending deformation, while the plate exhibits an unsymmetric bending deformation along the diagonal line in the fourth mode. Therefore, the common peak in the frequency response function of the bi-stable plate that is predicted by the analytical model, FEA and the experiment corresponds to the third resonance mode. In FEA, the model is ideal and symmetric, and thus the other modes are not excited in the time domain analysis. Since a symmetric

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**Table 1**

<table>
<thead>
<tr>
<th>Material properties of CCF300/5428 prepreg.</th>
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<tbody>
<tr>
<td>CFRP</td>
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<tr>
<td>( a_{11} = 0.4 \times 10^{-6} / \text{°C} )</td>
</tr>
<tr>
<td>( a_{22} = 25 \times 10^{-6} / \text{°C} )</td>
</tr>
<tr>
<td>Thickness = 0.125 mm</td>
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</table>
displacement function is employed in the analytical analysis, the bi-stable plate is unable to deform in mode 1, mode 2, and mode 4, that are shown in Fig. 11. However, in the experiment, the specimen is not ideal and has variabilities in its dimensions and symmetry, including the position of the inertial masses. Therefore, resonance responses occur with respect to the extra vibrational modes, other than mode 3, which were also excited and observed in the experiments. It should be noted that the predicted amplitude-frequency trend of the single well vibration of the bi-stable plate also exhibits nonlinear characteristics, given by a softening stiffness characteristic with increasing vibration amplitude.

The predicted and measured displacement of the plate corner for the 100 mm × 100 mm, [02/902]T and 100 mm × 100 mm, [03/903]T plates are presented in Fig. 12. The dynamic responses predicted by the analytical model agree well with experimental results, and the oscillation curves at the plate corner are almost perfectly sinusoidal.

8.4. Cross-well vibration

When the vibration amplitude of the bi-stable plate is larger than a critical level, the dynamic snap-through phenomenon occurs. In other words, the bi-stable plate transforms between the two stable configurations during the vibration. In the present study, if the snap-through phenomenon occurs in each vibration cycle, the vibration pattern is termed continuous cross-well vibration; otherwise, the vibration is termed intermittent cross-well oscillation.

In the experiment, intermittent cross-well oscillation of the 100 mm × 100 mm, [03/903]T plate was observed. When the excitation force was increased slowly and achieved a critical value, the vibration pattern of the 100 mm × 100 mm, [03/903]T plate suddenly transforms from the single-well oscillation to the intermittent cross-well oscillation. This phenomenon is clearly shown in Fig. 13, where the measured excitation acceleration is given as the amplitude increases slowly. During the intermittent cross-well vibration, the excitation is no longer sinusoidal and has irregular amplitude. This phenomenon indicates that the bi-stable plate and the shaker are strongly coupled during intermittent cross-well vibration. In this section, Eq. (22) is solved to analytically predict the cross-well vibration of bi-stable plates. In Eq. (22), the excitation displacement $w_0(t)$ is assumed to be unknown and the coupling between the plate and the shaker is considered.

The nonlinear dynamic response of the 100 mm × 100 mm, [03/903]T plate predicted by the analytical model is shown in Fig. 14. The amplitude of the sinusoidal excitation force of the shaker varies linearly with time, that is $F(t) = t \times \sin(44 \times 2\pi t)$. With the increase in the excitation amplitude, the vibration pattern of the bi-stable plate transforms from a single-well oscillation to an intermittent cross-well oscillation, and then to a continuous cross-well oscillation. The analytical model successfully predicts the irregular pattern of the intermittent cross-well vibration at the plate centre, which is also roughly consistent with the nonlinear characteristics observed in the experiment. When the continuous cross-well vibration occurs, the dynamic response of the bi-stable plate becomes regular again.

The measured and predicted corner displacements of the
100 mm × 100 mm, [0°/90°]T and 100 mm × 100 mm, [0°/90°]T plates corresponding to intermittent cross-well vibration are compared in Fig. 15. Although the predicted displacements do not match exactly the experimental results, the chaotic characteristics of the intermittent cross-well vibration are very close.

Fig. 16 presents the predicted corner displacements of the 100 mm × 100 mm, [0°/90°]T and 100 mm × 100 mm, [0°/90°]T plates during the continuous cross-well vibration. The excitation force of the shaker is 6 N. The continuous cross-well vibrations of these two bi-stable plates also exhibit slight nonlinearity; nevertheless, the cross-well vibration represented by the corner displacements is regular. The FFT analysis for the cross-well vibration of these two bi-stable plates are performed and shown in Fig. 17. The FFT results indicate the presence of the third harmonic. In the experiment, it was difficult to observe the continuous cross-well vibration, as the deflections or dimension asymmetry of the bi-stable plates had an adverse effect on the continuous snap-through event.

The dynamic response of the coupled system of the plate and the shaker does not directly reflect the natural dynamics of the bi-stable plates, whereas the natural dynamics can be obtained by exciting the uncoupled plate with a given displacement input. Specifically, Eq. (18) is applied to study the dynamics of bi-stable plates analytically, since the excitation acceleration of the plate is not coupled with the shaker in the dynamic model represented by Eq. (18). The time domain responses of the [0°/90°]T and [0°/90°]T plates excited by the sinusoidal input \( w_0(t) \) are predicted by Eq. (18), and the resulting vibration displacements at the plate corners are plotted in Fig. 18. The excitation amplitude increases by 0.1 g in every vibration cycle. The excitation amplitude is linearly varying with time, and the formula for the excitation input is also given in the caption of Fig. 18. The vibration curves in

![Graph](image)

Fig. 18. FFT analysis of the predicted free vibration of the [0°/90°]T plate in Fig. 7.

![Graph](image)

Fig. 19. Predicted free vibration of the 100 mm × 100 mm, [0°/90°]T plate with the proportional damping coefficient \( \alpha = 100 \).
Fig. 18 clearly show that when the excitation amplitude increases to a critical value, the vibration pattern of both plates transfer from the linear sinusoidal vibration to the cross-well vibration. The critical excitation acceleration amplitude, with which the vibration pattern transforms to the cross-well oscillation, for \([02/902]_T\) plate at 22 Hz is 3.6 g, and for \([03/903]_T\) plate at 35 Hz is 3.8 g. It is therefore concluded that the analytical model of Eq. (18) is capable to capture the characteristics of the chaotic snap-through and continuous snap through events.

The dynamic response of 100 mm \(\times\) 100 mm, \([02/902]_T\) and \([03/903]_T\) plates at different frequencies are predicted by both the analytical model given by Eq. (18), and FEA. A constant damping coefficient of \(\alpha = 100\) is adopted. Sinusoidal excitations with linearly increased amplitude are applied, that is \(w(t) = 2\pi \times 9.8 \times \sin(2\pi \times f \times t)\). Critical
Excitation amplitudes to induce cross-well vibration are determined according to the vibration curve of the plate corner, as illustrated in Fig. 18.

The critical excitation amplitudes at different excitation frequencies are predicted by both the analytical model and the FEA, and are compared in Fig. 19. For both plates, the analytical and numerical methods predict identical varying rate of critical excitation acceleration, and the lowest value. The overall curves given by Eq. (18) are slightly shifted to the right side compared with the FEA curves. The analytical results presented in Fig. 19 indicate that Eq. (18) slightly overestimates the stiffness of the plates. Nevertheless, the results demonstrate that the analytical model is able to accurately predict the nonlinearity of cross-ply bi-stable plates.

8.5. Degree-of-freedom of the analytical model and its efficiency

In the proposed analytical model, there are 17 terms in the assumed forms of \( w(t) \), \( \varepsilon_x^m(t) \) and \( \varepsilon_y^m(t) \), as presented in Eqs. (9) and (10). The sixth order out-of-plane displacement \( w(t) \) is assumed first. In order to test the minimum terms that are required to obtain accurate results, the forms of \( \varepsilon_x^m(t) \) and \( \varepsilon_y^m(t) \) are initially expanded to complete polynomials. Since the aim is to minimize the unknowns in the analytical model, the terms in \( \varepsilon_x^m(t) \) and \( \varepsilon_y^m(t) \) are deleted step by step in the testing simulation. After many attempts, it was found that each term left in Eq. (10) is critical and should remain to obtain accurate results. It was also interesting to note that, in Eq. (10), \( \varepsilon_x^m(t) \) has identical terms in \( x \) as \( w(t) \), and \( \varepsilon_y^m(t) \) has identical terms in \( x \) as \( w(t) \).

For illustration, the static load–displacement curves of the 100 mm × 100 mm, \([0\_90\_90]_T \) plate predicted using different number of terms in the analytical model are compared. The numerical results presented in Fig. 20. Concentrated loads are applied on the corners of the plate, which is mounted at the central point. For the 15 degrees-of-freedom analytical model, the \( c_3(t)x^2y^3 \) and \( d_4(t)x^3y^2 \) terms in \( \varepsilon_x^m(t) \) and \( \varepsilon_y^m(t) \), i.e. Eq. (10), are deleted. The \( c_1(t)y^5 \) and \( d_4(t)x^3 \) terms are further deleted to reduce the number of degrees-of-freedom to 13. The 13, 15 and 17 degrees-of-freedom in the analytical models have similar precision if the applied loads are smaller or larger than the snap-through load. However, compared to the FEA results, the prediction errors given by the analytical model at the snap through load are increased when the number of degrees-of-freedom is reduced. As the snap through phenomenon occurs in cross-well vibration, inaccurate prediction of the snap through will lead to inaccurate dynamic analysis for the bistable plates. In contrast, the 17 degrees-of-freedom analytical model predicts accurate snap through phenomenon of bistable plates, and is also validated by the FEA results. This numerical simulation demonstrates that using 17 terms is the minimum number of degrees-of-freedom for the analytical model proposed in this work.

In this study, the “NDSolve” function in Mathematica™ is applied to solve the nonlinear differential equations, i.e. Eqs. (18) and (22). Each of the time domain results presented in Figs. 7–18 only requires less than one minute calculation time using the proposed analytical model. In contrast, the time-domain dynamic analysis using the finite element method is very time-consuming. To obtain more accurate results, the time step in the “Implicit, Dynamic” step should be sufficiently small. In this study, the time step is set to be 0.0025 s, therefore a set of numerical results in time domain of 1 s needs 2000 iterations. A cross-well vibration lasting for 2 s usually takes 20–40 min, depending on the mesh size. In Fig. 19, the prediction of the critical excitation acceleration by FEA requires much more computation time than the analytical model. This comparison of computational time demonstrates the high efficiency provided by the analytical model.

9. Conclusion

The dynamics of bi-stable plates are complicated due to the nonlinearity and the snap-through phenomena. Although several dynamic models have been reported, either the dynamic snap-through load was overestimated [24] or the dynamic response at single points were predicted [22,23]. An analytical dynamic model with only 17 unknown terms is established based on Rayleigh’s method and Hamilton’s principle in this study. This established model can accurately predict the nonlinear dynamic response of bistable plates. Moreover, this analytical model is able to predict the nonlinear dynamic response over the entire region of bi-stable plates, rather than just few specific points. The influence of the inertial mass and the coupling between a bi-stable plate and the electrical shaker is also considered in the dynamic model. The free vibration, excited vibrations including single-well, intermittent cross-well and continuous cross-well, are predicted and studied. The predicted results of the proposed analytical model for different vibration patterns of bi-stable plates coincide very well with FEA results and experimental results.
Fig. 15. Displacements of the cross-ply bi-stable plates corresponding to intermittent cross-well oscillation.

(a) 100mm × 100mm, [0°/90°]Τ

(b) 100mm × 100mm, [0°/90°]Τ
Fig. 16. Displacement curves of cross-ply bi-stable plate corresponding to cross-well oscillation (dT = 160 °C, C = 5).

(a) 100mm × 100mm, [0₂/9₀₂]ₜ

(b) 100mm × 100mm, [₀₃/9₀₃]ₜ

Fig. 17. FFT of the corner displacements in Fig. 16.

(a) 100mm × 100mm, [₀₂/9₀₂]ₜ, analytical

(b) 100mm × 100mm, [₀₂/9₀₂]ₜ, FEM

(c) 100mm × 100mm, [₀₃/9₀₃]ₜ, analytical

(d) 100mm × 100mm, [₀₃/9₀₃]ₜ, FEM
The proposed reduced order analytical model for the nonlinear dynamic analysis of composite plates possesses several advantages. Firstly, this proposed analytical model provides a very efficient and effective means to predict the nonlinear dynamics for cross-ply composite bi-stable plates. Secondly, this analytical model is very simple to implement, provided that the material properties and dimensions are provided. Thirdly, the procedure to predict the nonlinear dynamics of bi-stable plates using this analytical model is a relatively simple extension to the static analysis. With only 17 unknown terms, the analytical model is not sophisticated enough to perform very complicated dynamical analysis for bi-stable plates. Nevertheless, an extension of this work to a more general model for angle-ply bi-stable composite plates is straightforward.

Fig. 18. The time domain responses of bi-stable plates excited by a sinusoidal input, predicted by analytical model, i.e. Eq. (18). The excitation amplitude increases by 0.1 g in each cycle, excitation acceleration is \( \ddot{w}_\theta(t) = f \cdot t \times 0.1g \times \sin(2\pi \cdot t) \).

Fig. 19. Critical excitation acceleration needed to induce the cross-well vibrations for the 100 mm \( \times \) 100 mm, \([0_{2/90}]_1\) and \([0_{3/90}]_1\) plates.

Fig. 20. Static load-displacement curves of the 100 mm \( \times \) 100 mm, \([0_{2/90}]_1\) plate.
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Appendix A

\[ u''(t) = \left( c(t) + A^{-1}N_x^T \right) x - \frac{2}{3} a(t) x^3 - \frac{8}{5} a(t) a(t) x^5 - \frac{1}{7} (8a(t) x^2 + 12 a(t) a(t) x^2) x y - \frac{8}{5} a(t) a(t) x^6 + \frac{1}{11} (8a(t) x^2 + 12 a(t) a(t) x^2) x y^2 - \frac{18}{11} a(t) x^2 + c(t) x y + c(t) x^6 + c(t) x y^6 \\ + \frac{1}{3} (c(t) a(t) - 4a(t) x(t)) x y^2 - \frac{2}{3} (c(t) a(t) - 4a(t) x(t)) x y^4 - \frac{8}{5} a(t) a(t) x^6 - \frac{12}{7} a(t) a(t) x^7 - 2a(t) B_{11} + b(t) B_{12} x - \left( 4a(t) B_{11} + \frac{2}{3} c(t) B_{12} \right) x y^2 - 6a(t) B_{12} x^5 \\ - (12a(t) B_{12} + 2c(t) B_{12}) x y^2 - 30a(t) B_{12} x y^4 \] \hspace{1cm} (A.01)

\[ v''(t) = \left( d(t) + A^{-1}N_y^T \right) y - \frac{2}{3} b(t) y^3 - \frac{8}{5} b(t) b(t) y^5 - \frac{1}{7} (8b(t) y^2 + 12 b(t) b(t) y^2) y^7 - \frac{8}{5} b(t) b(t) y^6 - \frac{18}{11} b(t) y^2 + d(t) x y^2 + d(t) x y^4 + d(t) x y^6 \\ + \frac{1}{3} (d(t) a(t) - 4a(t) y(t)) x y^2 - \frac{2}{3} (d(t) a(t) - 4a(t) y(t)) x y^4 - \frac{8}{5} b(t) b(t) y^6 - \frac{12}{7} b(t) b(t) y^7 + 2b(t) B_{11} + a(t) B_{12} y + \left( 4b(t) B_{11} + \frac{2}{3} c(t) B_{12} \right) y^2 - 6b(t) B_{12} y^5 \\ + (12b(t) B_{12} + 2c(t) B_{12}) y^2 + 30b(t) B_{12} y^4 \] \hspace{1cm} (A.02)

\[ y''(t) = \left( 4a(t) b(t) + 2c(t) + 2d(t) \right) x x y + \left( \frac{2}{3} c(t) + 4d(t) + 8a(t) b(t) + \frac{4}{3} a(t) e(t) \right) x y^2 + \left( 2 d(t) + 4c(t) + 8a(t) b(t) + \frac{4}{3} b(t) e(t) \right) x y^3 \\ + \left( 16a(t) b(t) - \frac{4}{3} e(t) \right) x y^4 + \left( 6d(t) + 12a(t) b(t) + \frac{24}{5} a(t) e(t) \right) x y^5 + \left( 6c(t) + 12a(t) b(t) + \frac{24}{5} b(t) e(t) \right) x y^6 + 24a(t) b(t) x y^7 + 36a(t) b(t) x y^8 + \frac{60}{7} b(t) e(t) x y^9 + \frac{60}{7} a(t) e(t) x y^10 \] \hspace{1cm} (A.03)

\[ M_i = \rho \int \int \int \left( \frac{\partial \sigma_{ii}}{\partial x} \frac{\partial \sigma_{ij}}{\partial x} \right) dx dy d i, j = 1 \sim 17 \] \hspace{1cm} (A.04)

\[ K_i(X_i) = \left\{ \frac{\delta (P_i)_{plan} - W_i(t)}{\delta X_i(t)} \right\}, \quad i = 1 \sim 17 \] \hspace{1cm} (A.05)

\[ G = -\rho \int \int \int \left( \frac{\partial \omega_t}{\partial X_i(t)} \right) dx dy, \quad i = 1 \sim 17 \] \hspace{1cm} (A.06)

\[ M_i = \left( \begin{array}{ccc} m & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \rho \int \int \int \left( \frac{1}{\delta x} \frac{\partial \omega_t}{\partial x} \right) dx dy d i, j = 2 \sim 18 \] \hspace{1cm} (A.07)

\[ M_{inertial1} = 4 m_{inertial} \left( \frac{\partial \omega_t}{\partial x} \right) d i, j = 1 \sim 17 \] \hspace{1cm} (A.08)

\[ M_{inertial2} = 4 m_{inertial} \left( \frac{\partial \omega_t}{\partial x} \right) d i, j = 2 \sim 18 \] \hspace{1cm} (A.09)

\[ F(X_i)_{inertial} = -4 m_{inertial} \frac{d^2 (\omega_t)}{dt^2} \left( \frac{\partial \omega_t}{\partial X_i(t)} \right), \quad i = 1 \sim 17 \] \hspace{1cm} (A.10)

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.compstruct.2018.06.072.

Reference


