Parameter Subset Selection in Damage Location

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ABSTRACT

Methods to locate damage in structures, using a finite element model and low frequency measured vibration data, have attracted considerable interest. A large number of parameters are required to ensure that the damage location and mechanism may be modelled by at least one set of parameter values. Generally the identified parameter values are not unique and extra information must be incorporated into the identification. The finite element model of a damaged structure is likely to be in error at only a small number of sites. This is equivalent to requiring that only a subset of parameters are in error, and leads to the methods of subset selection. The standard method uses the sensitivity matrix based on the initial finite element model to choose the parameter subset. Many residuals used for damage location are non-linear functions of the parameters, and this paper examines the relationship between the subset selection and the iteration required for the parameter estimation. Also measurements are often taken periodically and it is necessary to trend the changes in the important parameters. This requires that the best parameter subset is chosen based on multiple data sets. Several strategies to deal with the required iteration and multiple data sets are outlined and the methods are tested on a simulated cantilever beam, both with and without systematic errors.
1. INTRODUCTION

Recently, there has been a considerable demand for more accurate techniques to detect and locate damage using measured vibration data, particularly in large space structures. Doebling et al. [1] gave an extensive survey of the field. Friswell and Penny [2] summarised many of the difficulties common to all damage location methods. Damage will cause the stiffness distribution in the structure to change which may be detected by measuring its dynamic response. Two distinct philosophies have been considered to locate the damage in a structure using measured vibration data. The first group of methods use techniques of finite element model updating [3, 4] to obtain a corrected set of physical parameters or a corrected stiffness matrix that reproduces the measured data. The alternative group of methods assume a candidate set of possible damage scenarios which include both the damage mechanism and the location. The damage scenarios that best fit the measured data are then chosen, often based on statistical arguments [5-8]. These methods will now be reviewed briefly.

Early papers in model updating changed complete mass and/or stiffness matrices. The goal was to reproduce the measured data (usually the modal model), by changing the stiffness matrix (for example) as little as possible in some minimum norm sense. A number of problems exist with these methods. There is no guarantee that the resulting matrices are positive definite (or semi-definite for structures with free-free modes), and extra modes may be introduced into the frequency range of interest. More serious is that forcing the model to reproduce the data doesn’t allow for the errors that will be present in the measured data. Mode shapes, in particular, can only be measured with a limited accuracy. The major problem in the application of these methods to damage location, and indeed to error location in model updating, is that all the elements in the matrices may be changed. If only a small number of sites are modelled incorrectly (or are damaged) then only a small number of the matrix elements will be changed. Generally, because of the minimum norm optimisation in the updating method, all the matrix elements would be changed a little, rather than a small number of elements changed substantially. Thus the effect of any damage present would be spread over all the degrees of freedom making location difficult. Kaouk and Zimmerman [9] tried to overcome this problem by ensuring that the change in the stiffness matrix was low rank. Doebling [10] extended this approach to retain the original physical connectivity of the structure and thus ensure that new load paths are not introduced. This does not ensure that the change in stiffness will be local, as the stiffness change could be global but low rank. Also the method, as
proposed, requires the rank of the stiffness change to be equal to the number of modes measured, and the localisation of damage often becomes worst as more measured modes are included.

Most common methods in model updating rely on a parametric model of the structure [3, 4]. The measured data may be in the form of frequency response function data, or in terms of the modal data. The updating exercise minimises residuals based on the modal quantities, or the input (equation) error or output error in the frequency domain. In model updating, the number of parameters may be reduced by including only those parameters that are likely to be in error. This option is not available for damage location since the site of the damage is unknown a priori. Thus a large number of candidate parameters must be included and the parameters cannot be identified uniquely. In this situation all the parameters are likely to be changed, and regularisation must be applied to generate a unique solution. Regularisation generally applies extra constraints to the parameter estimation problem to ensure a unique solution [11-15]. Most regularisation techniques rely on minimum norm type solutions that will tend to spread the identified damage over a large number of parameters, making damage location difficult.

The major difference between model updating and damage/error location is that in damage location only a limited number of parameters are likely to be in error. If only these parameters are chosen then the parameter estimation problem would be over-determined. Unfortunately we do not know which parameters might be in error. Determining which parameters are in error may be thought of as a form of regularisation known as subset selection. Another group of methods, very similar to subset selection, assume a candidate set of possible damage scenarios which include both the damage mechanism and the location. The change in dynamic response due to the damage is predicted, usually in terms of the natural frequencies. The predicted change in the response of the structure for all the damage scenarios is then compared to the measured change and the closest damage case is chosen. Statistical methods are often used to identify the most likely location of the damage. Cawley and Adams [6-8] used this type of method and found that damage in specimens fabricated from composite materials could be detected. Friswell et al. [5] used a similar method but also obtained an indication of the quality of the predictions.

In this paper subset selection is applied to the damage location problem. The finite element model is parameterised so that all the likely damage scenarios are modelled by
some combination of the parameter values. The methods detailed in this paper are then used to pick out a limited number of these parameters that produce the 'best' fit to the measured data. For example, if the location of damage in a space frame structure is required, then the complete set of parameters might consist of the stiffness of the constituent spars. It is expected that only one or two of these spars might be damaged, and the aim is to identify the corresponding parameters by the following procedures. Of course, there might be other errors in the undamaged model, and the damage location method should be robust so that the correct damage sites are identified despite these errors.

Subset selection is well established in the statistical literature [16], and is based on a sub-optimal choice of parameters to reduce the computational burden. The selection of a suitable subset of parameters for estimation has been used for error location in structural dynamics [17], for actuator location [18] and for damage detection [19, 20]. This paper extends these methods in two ways; by considering the effect of approximating the non-linear estimation problem by its linearised equivalent and by extending the method to multiple measurement data sets. Section 2 outlines some of the difficulties for damage location methods arising from the measurement noise and modelling errors. The problem of damage location is stated formally in Section 3 and the standard method of subset selection is introduced in Section 4. The strategies that may be employed to produce sub-optimal subset selections for the non-linear parameter estimation problem are considered in Section 5. The basic method of subset selection is extended to multiple data sets in Section 6 by minimising angles between subspaces, a concept that follows naturally from standard subset selection that minimises angles between vectors. Finally a cantilever beam example is used in Section 7 to demonstrate the properties of the methods.

2. MEASUREMENT NOISE AND MODEL ERRORS

The comments above have indicated some of the problems with damage identification. There will always be errors in the measured data and the numerical model that affect all the algorithms. These errors, and the adequacy of the data, are now discussed.

One of the major problems in damage location is the reliance on the finite element model. This model is also an important strength because the very incomplete set of measured data requires extra information from the model to be able to identify the damage location.
There will undoubtedly be errors even in the model of the undamaged structure. Thus if the measurements on the damaged structure are used to identify damage locations, the methods will have great difficulty in distinguishing between the actual damage sites and the location of errors in the original model. If suitable parameters are not included to allow for the undamaged model errors then the result will be a systematic error between the model and the data. Identification schemes generally have considerable difficulty with systematic errors. It is very likely that the original errors in the model will produce frequency changes that are far greater than those produced by the damage. There are two approaches to reducing this problem, although both rely on having measured data from an undamaged structure. The first is to update the finite element model of the undamaged structure to produce a reliable model [3, 4]. Obviously the quality of the damage location assessment is critically dependent upon the updated model being physically meaningful. Generally, this requires model validation using a control set of data not used for the updating. The second alternative uses differences between the damaged and undamaged response data in the damage location algorithm. To first order, any error in the undamaged model of the structure that is also present in the damaged structure will be removed. This does rely on the structure not changing, except for the damage, between the two sets of measurements.

The range of frequencies employed in damage location has a great influence on the resolution of the results and also the physical range of application. The great advantage in using low frequency vibration measurements is that the low frequency modes are generally global and so the vibration sensors may be mounted remotely from the damage site and fewer sensors may be used. The problem with low frequency modes are that the spatial wavelengths of the modes are large, and typically are far larger than the extent of the damage. The spatial resolution of the damage identification scheme requires that there is a significant change in the response due to the damage between two adjacent potential damage sites. If low frequency modes are used then this resolution is closely related to the spatial wavelengths of the modes.

One very difficult aspect of damage assessment is the change in the measured data due to environmental effects. This is one undesirable non-stationary effect and makes damage location difficult. Of course progressive damage is also a non-stationary phenomena, but if other non-stationary effects dominate the changes due to the damage then obviously the latter will be difficult to identify. The changes due to environmental effects may be modelled using a relatively small number of parameters. Hence the subset selection
method should identify the environmental parameters that change together with the parameters that model the damage.

3. PROBLEM STATEMENT

The possible damage locations and mechanisms are represented by parameters of the finite element model, and assembled into a vector $\theta$. The parameters consist of perturbations from the modelled quantities so that a zero parameter value corresponds to no damage. Measurements are taken on the structure and assembled into a vector $z_m$. The parameters should be changed to improve the correlation between the predictions and the measurements. Fritzen [21] considered a range of measurement residuals. The examples in this paper will use only natural frequencies, although the methods can use other measured data without any problem. The damage location problem is to find the subset of the unknown parameters that is best able to replicate the measured data. Formally, this may be stated as: find the parameter subset that minimises the penalty function

$$ J = \left\| z_m - z(\theta) \right\|^2 $$  \hspace{1cm} (1)

where $z(\theta)$ is a non-linear function of the parameters that represents the predictions from the model, in the same form as the measurements. The minimum number of parameters are required to produce a sufficiently small residual in Equation (1). Adding an extra parameter to a subset will always reduce the residual and if the only requirement was a small residual then a large subset would be chosen. Section 4.2 considers the problem of choosing the number of parameters in the subset in more detail.

Equation (1) represents a non-linear optimisation problem that would require an enormous computation effort to consider each parameter subset in turn. To enable progress to be made equation (1) is linearised, and sub-optimal methods are used to select the best subset of the parameters. The measured data may be written in terms of a first order Taylor Series in the parameters as

$$ z_m = z(\theta) + S \theta + \text{higher order terms} $$ \hspace{1cm} (2)

where $S$ is the sensitivity matrix and contains the first order derivatives of the measured quantities with respect to the parameters. The calculation of these derivatives for
eigenvalues and eigenvectors is standard [22-24] and is relatively straight-forward for most residuals. Neglecting the higher order terms, and rearranging equation (2) gives

$$S \theta = z_m - z(0).$$

(3)

Differences between measurements taken on the damaged and undamaged structures may also be used, as discussed in Section 2. Thus, if $z_u$ represents the measurements on the undamaged structure and $z_d$ represents the measurements on the damaged structure, the equations of interest are

$$S \theta = z_d - z_u$$

(4)

Usually the equations will be weighted. For example, if natural frequencies were used directly in equations (3) or (4) then the higher frequencies will effectively be weighted more highly. Using relative (or percentage) changes in the natural frequencies is a good solution. The relative weighting between natural frequencies and mode shapes should recognise the relative uncertainty in each measurement.

Equations (3) and (4) are both linear in the parameters. The linear subset selection problem may now be posed as follows: given the set of $n$ equations in the $p$ parameters,

$$A \theta = b$$

(5)

where $A$ is an $n \times p$ matrix and $b$ is a vector of length $n$, find the subset of the parameters that minimises the residuals in these equations. It is implicit in this statement of the problem that the minimum number of parameters are required that produce a sufficiently small residual. Typically, for damage location and error localisation in model updating, there are many more parameters than measurements ($p > n$). Although using mode shapes or frequency response function data it may be possible to produce an over-specified set of equations, in general the solution to these equations will be ill-conditioned. Physically this means that there is insufficient information in the measured data to identify all the parameters.

Very often, modal tests on a structure will produce different results under different conditions or at different times due to environmental changes, different levels of forcing, changes in material properties over time and the occurrence of damage. For each set of measurements an equation equivalent to equations (3) or (4) may be generated, with the
same sensitivity matrix. These sets of equations may be assembled into an equation of the same form as equation (5), but where θ and b are now matrices. Usually the changes in measurements will not span a complete space and the subspace containing the sets of measurements may have a small dimension, compared to the number of measured data sets. For example, a set of measurements made on a bridge might be found to depend on humidity and temperature in which case the subspace of measurements would contain only two independent measurement vectors. Thus we wish to determine which parameters are the most important to model the changes in all the measurement sets. This problem will be considered further in Section 6.

4. PARAMETER SUBSET SELECTION

Millar [16] gave a good summary of the subset selection method for the single measurement case. The important aspects for the application to structural dynamics will be outlined briefly. The emphasis in this paper will be on the forward selection method, where parameters are selected one at a time. Other selection strategies such as backward selection, where parameters are eliminated one at a time, or sequential replacement algorithms [16] have limited value in damage location [19]. Millar considered the statistical properties of the estimators, including the bias in the parameter estimates due to the selection procedure and due to the omission of parameters (named selection bias and omission bias). Millar also discusses in depth the trade-off between reduced bias but increased variance in the parameter estimates as the number of parameters is increased.

4.1. Forward Selection

The approach adopted in this paper is to assume that only a subset of the parameters in equation (5) are non-zero. The optimum subset is chosen that minimises the residual in equation (5). Lallement and Piranda [17] used an iterative procedure to produce a sub-optimal solution, commonly known as forward selection. Among the columns of A, the single column is sought which best represents the vector b. If the columns of A are given by \( a_j \), so that \( A = [a_1 \ a_2 \ldots \ a_p] \), then the selected parameter is that which minimises the residual

\[
J = \|b - a_j \hat{\theta}_j\|^2
\]

where \( \hat{\theta}_j \) is the least squares estimate of the \( j \)th parameter, and is
\[ \hat{\theta}_j = a_j^T b / a_j^T a_j \]  

(7)

Expanding equation (6), using equation (7), gives

\[ J = b^T b - (a_j^T b)^2 / a_j^T a_j \]

(8)

and the \( j \)th parameter is selected that maximises

\[ (a_j^T b)^2 / a_j^T a_j \]

Equivalently, since \( b \) is a constant vector,

\[ \cos^2 \phi_j = \frac{(a_j^T b)^2}{(a_j^T a_j)(b^T b)} \]

(9)

is maximised where \( \phi_j \) is the angle between the vectors \( b \) and \( a_j \). Thus the \( j \)th parameter is selected that minimises the angle between the vectors \( b \) and \( a_j \).

Next the combination of two columns of \( A \) which constitutes the best sub-basis for the representation of \( b \) is determined. In the forward selection procedure the second parameter is obtained in a sub-optimal way, by retaining the first parameter selected. Let \( j_1 \) represent the first parameter selected and let the corresponding column of \( A \) be \( a_{j_1} \). The optimum value for the \( j_1 \)th parameter is then

\[ \hat{\theta}_{j_1} = a_{j_1}^T b / a_{j_1}^T a_{j_1} \].

(10)

The vector \( b - a_{j_1} \hat{\theta}_{j_1} \) is then orthogonal to \( a_{j_1} \) and the subspace spanned by the columns of \( A \) that is orthogonal to \( a_{j_1} \) must be searched. Thus the columns of \( A \) and the vector \( b \) are replaced with

\[ a_j \rightarrow a_j - a_{j_1} \hat{\alpha}_j \quad \text{where} \quad \hat{\alpha}_j = a_{j_1}^T a_j / a_{j_1}^T a_{j_1} \]

\[ b \rightarrow b - a_{j_1} \hat{\theta}_{j_1} \]

(11)

The procedure is now repeated on this reduced problem, to find the parameter \( \theta_j \), for \( j \neq j_1 \), that gives the smallest residual, or equivalently the smallest angle \( \phi_j \).
An iterative process is then produced. Let $m$ be the number of parameters selected. First, with $m=1$ the single parameter that is best able to represent the data is selected. Subsequent iterations retain the parameters chosen in previous steps and select the parameter from those remaining that, together with those already chosen, is best able to represent the data. At each value of $m$ this represents a one dimensional optimisation, rather than an $m$ dimensional optimisation in the general case, but the method is sub-optimal.

### 4.2. Efroymson’s Algorithm

The algorithm proposed by Efroymson [16, 25] is a variation on the forward selection, but allows for selected parameters to be deleted. Suppose that $m$ is the number of parameters selected, given by $\{\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_m}\}$. Then the residual sum of squares, $J_m$, is given by

$$J_m = \left( b - \sum_{i=1}^{m} a_{ji} \hat{\theta}_{ji} \right)^2$$

(12)

where $\hat{\theta}_{ji}$ is the least squares estimate of the $j$th parameter, using only the $m$ selected parameters. For each parameter in the selected subset, the residual sum of squares using just the remaining $m-1$ parameters is calculated. The test ratio, $R_d$, based on the minimum of these sums, $J_{m-1}$, is calculated as

$$R_d = \frac{J_{m-1} - J_m}{J_m/(n-m-1)}$$

(13)

and compared to a predetermined value. If $R_d$ is less than this value the corresponding parameter is eliminated from the selected subset.

Efroymson also determined a ratio to decide whether a parameter should be added to the selected subset. If $J_{m+1}$ is the minimum residual sum of squares that may be obtained by adding the parameter $\theta_{j_{m+1}}$ to the original set $\{\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_m}\}$, then if the ratio

$$R_a = \frac{J_m - J_{m+1}}{J_{m+1}/(n-m-2)}$$

(14)

is greater than a predetermined value the parameter $\theta_{j_{m+1}}$ is added to the selected set. The ratio $R_a$ will be used in the example as a guide to determine the optimum number of...
parameters to use. If the value of $R_a$ reduces substantially then the corresponding parameter is not added to the subset, and this initial subset is taken as the optimum.

5. ITERATION AND SELECTION STRATEGIES

As in the linear case, an exhaustive search of all possible parameter selections to find the lowest residual in equation (1), is computational prohibitive. Sub-optimal algorithms must therefore be derived. These may be based on the selections from the linearised equations of the standard method. There are a number of possible strategies determined by the ordering of iteration and selection. All of these are based on retaining a number of the best performing subsets throughout the process, which may therefore produce subsets that are chosen more than once. These will now be outlined in order of increasing computational requirements.

5.1. The Standard Method

The standard method using the sensitivity matrix derived from the analytical model may be applied and a relatively small number of the best parameters retained at each stage. Suppose $n_s$ parameters are selected at each of $m$ stages. For example, if $n_s = 2$ then we might choose parameters 5 and 9 as the best two parameters which on their own are best able to represent the measured data. Then we retain parameter 5 and look for the parameter that combined with parameter 5, yields the best match. We retain the best 2 and then repeat the process with parameter 9, and so on. The result will be a total of $n_s$ subsets of size 1, $n_s^2$ subsets of size 2, $n_s^3$ subsets of size 3, and so on up to $n_s^m$ subsets of size $m$. The parameter subsets have been selected using the sensitivity matrix based on the initial finite element model. For each of these subsets the parameters are updated until convergence. The size of the residuals based on the updated parameters may then be compared and the subset with the lowest residual chosen.

5.2. Optimising Already Chosen Parameters

An alternative strategy is to iterate until convergence at every stage based on the parameters selected during the previous stages. Thus at the first stage the parameters are chosen as before. The optimum value for the selected parameter is then calculated by iteration. Using the sensitivity matrix based the converged parameter a second parameter is selected. Note that this sensitivity matrix must be orthogonalised with respect to the column of the converged sensitivity matrix corresponding to the chosen parameter. The
optimum values of the two parameters are then obtained by iteration. Note that both of the chosen parameters are optimised together, and in general the value of the first parameter will not be equal to its value at the first stage. This process continues with subsequent stages. As in the standard method more than one selection should be retained at each stage to produce a number of parameter subsets.

5.3. Iteration at Every Stage

The final strategy is to iterate before the best parameters are chosen. Thus, at the first stage, the optimum value for each candidate parameter is obtained by iteration. The values of the residuals based on these converged parameter estimates are compared and the best parameter is chosen. This parameter is retained and the second stage considers each remaining candidate parameter in turn. The optimum values of the parameters (the one already selected and the candidate parameter) are obtained by iteration, and the best parameter chosen based on the lowest residual. Once again, more than one selection should be retained at each stage to produce a number of parameter sets.

6. SUBSET SELECTION USING ANGLES BETWEEN SUBSPACES

The concept of angles between subspaces is a generalisation of the concept of angles between vectors used earlier. In three dimensions, it is easy to visualise the angle between a pair of lines, that is subspaces of dimension one, or indeed the angle between a line and a plane, that is subspaces of dimension one and two respectively. Bjorck and Golub [26] described the definition and calculation of the angles between subspaces. These ideas have been applied in structural dynamics in the areas of damage location [27], model updating using perturbed boundary condition testing [28], mode shape correlation [29] and sensor location [30]. The definition and calculation of the angles between subspaces will now be described.

Suppose the subspaces are represented by two matrices $F$ and $G$, of size $n \times m$ and $n \times q$ respectively. The corresponding subspaces are obtained from the columns of the matrices, that is from the range of $F$ and $G$, and for full rank matrices these subspaces have dimension $m$ and $q$ respectively. We can obtain an orthogonal basis for these subspaces using the QR algorithm thus,

$$ F = Q_F R_F \quad G = Q_G R_G $$

(15)
where \( Q_F \) and \( Q_G \) are orthogonal matrices of dimension \( n \times m \) and \( n \times q \) respectively and \( R_F \) and \( R_G \) are upper triangular. If \( q \leq m \) there will be \( q \) principal angles between the subspaces, \( \psi_i \), which are computed from the singular value decomposition of \( Q_F^T Q_G \). Thus

\[
\cos \psi_i = \sigma_i \left( Q_F^T Q_G \right)
\]  

(16)

where \( \sigma_i \) denotes the \( i \)th singular value. Because the matrices \( Q_F \) and \( Q_G \) are orthogonal all of the singular values are between zero and unity, and hence the inverse cosine is well defined. If any one of these angles is 90°, then there is one combination of the columns of \( F \) which is orthogonal to all of the columns of \( G \) and vice-versa. If any one of these angles is 0°, then there is one combination of the columns of \( F \) which is equal to some combination of the columns of \( G \) and vice-versa.

Thus, finding the angles between the subspaces makes it possible for an assessment to be made of how well the subspaces overlap. It is obviously appropriate to examine the worst angle from the point of view of acceptability of the fit of the parameters to the data. Thus the parameter is chosen that is best able to represent all the data sets.

If only one set of measured data is available then the application of subspace matching reduces to the scalar product, yielding the standard method for parameter selection. Section 2 outlined certain circumstances in which multiple sets of measurements might arise. Given multiple measurement sets, the parameter selection reduces to choosing a subset of the columns of the sensitivity matrix, \( S \) or \( A \) in the notation of equation (5), that best match the columns of the matrix \( B \), made up of vectors \( b_i \), corresponding to the vector \( b \) in equation (5). We may proceed in a similar manner to the forward selection procedure described earlier. After the selection of the first parameter, the subspace orthogonal to the columns of \( A \) corresponding to the selected parameters must be searched. The resulting method is similar to that given in equations (11), except there is now more than one vector \( b \).

7. A NUMERICAL EXAMPLE

The ideas contained in this paper will now be tested on a simulated cantilever beam example. Although somewhat artificial, all the important features and problems damage
location may be demonstrated. The ultimate test of any procedure is an experimental example, but the advantage of a simulated example is that damage may be introduced in a controlled and known way. To ensure that the conclusions from such simulations are valid the effects of noise and systematic errors must be incorporated [2]. The beam has a cross section of 25 mm x 50 mm, a length of 1 m and is assumed to be rigidly clamped at one end. Only motion in the plane of the thinner beam dimension is considered. The beam has a Young's modulus of 210 GN/m² and a mass density of 7800 kg/m³. The set of candidate parameters is chosen to be large and consists of one global parameter, the Young's modulus, two parameters per element, the element stiffness and the element mass, and two parameters per node, namely a discrete mass and a discrete spring. Forward selection of the best subset is used, based on the subspace angles. For the single measurement case with no iteration this corresponds to the standard best subspace approach [17]. Efroymson’s algorithm is only used to determine if a parameter should be added to the current selected parameter subset based on the \( R_a \) ratio of equation (14). A decision value for this parameter is not predetermined, but the relative values of this ratio as more parameters are added is used to determine how large the parameter subset should be. The measurement vector contains the first 8 natural frequencies of the beam, and the procedure is based on using the difference in these natural frequencies between the damaged and undamaged structures, equation (4). In all cases these frequency differences are weighted using the undamaged natural frequencies so that the ‘measurement’ vector in equation (5) contains relative natural frequency changes.

7.1. The Standard Method - No Errors

The first test of any method is its application to a simulated example with no noise or systematic errors. Any parameter changes in the model should be identified exactly. The simulated 'measurements' are consist of the relative changes in the first eight natural frequencies of the beam taken from a model with 20 elements. The undamaged natural frequencies are taken from the uniform beam, whilst the damaged frequencies are derived from a model where the stiffness of element 4 has been reduced by 30%. Table 1 gives the damaged and undamaged natural frequencies, showing that the 30% damage only results in a 2.4% change in natural frequency at most. These small frequency changes are typical in damage location and are one of the major difficulties in the identification of the location of damage [31]. Measurement noise, environmental factors and structure non-stationarity can easily lead to incorrect conclusions on damage location. The iteration strategy outlined in Section 5.1 is used.
Figure 1 shows the angles between the columns of the sensitivity matrix of the initial finite element model and the vector of the changes in the first 8 natural frequencies due to the damage, equation (9). Clearly the column relating to the stiffness of element 4 has a small angle, although it is not zero. It is not zero because equation (4) is only a first order approximation and the extent of the damage (30%) is large. For the four parameters with the lowest angle, Table 2 shows the residual at the first iteration and also after convergence based on optimising the single chosen parameter. Clearly the residual relating to the stiffness of element 4 goes to zero on convergence, showing that this parameter does indeed represent the damage location. Returning to Figure 1, changing the mass of element 17 is also able to model the 'measured' changes accurately. This problem arises because of the symmetry of the beam, and the fact than no spatial information is incorporated into the measurements. Mode shapes could also be incorporated into the measurement vector, although the accuracy with which they could be measured may be insufficient to show a change in the mode shapes due to damage. This is an example of the more general problem, where damage or changes in the parameters at more than one location causes the same changes in the lower natural frequencies.

What happens if we change more than one parameter? Suppose that a 0.1 kg mass is added to node 12, in addition to the 30% stiffness change to element 4. Table 3 shows the results when the best subsets of 1, 2 and 3 parameters are chosen. At each stage the 2 best parameters are chosen. The residuals listed in the parameter 1 column are the residuals when subsets of a single parameter are used. Those residuals in the parameter 2 column correspond to subsets of 2 parameters. Also shown are the residuals after convergence based on optimising the values of the chosen parameters. From the values of the test ratio $R_a$ and the residuals, it is clear that the two correct parameters should be selected. A number of other parameter subsets have small residuals and the addition of random noise may make the selection of the best subset more difficult.

### 7.2. The Standard Method - With Random Noise

Random noise will always be present in measurements, but large levels of random noise will cause the subset selection algorithms to choose incorrect parameters. To demonstrate the effect of random noise on the calculation of the angles between subspaces, the example of the last section, where only the stiffness of element 4 is reduced, will be used. To show the effect of random errors it is necessary to compute the statistics of the
calculated subspace angles, based on some distribution of measurement noise. Suppose that a random error with a Gaussian distribution and standard deviation of 0.1% is added to the ‘measured’ natural frequencies. Figure 2 shows the mean calculated angle, obtained from equation (9), and the standard deviation of this angle based on a Monte-Carlo type simulation with 10,000 samples. The mean is shown by the height of the bar, and the error bars denote +/- 1 standard deviation. The stiffness of element 4 and the mass of element 17 would still be chosen as the parameters that best model the damage. Figure 3 shows the effect of increasing the ‘measurement’ noise to 0.5%, and shows that it would be more difficult to locate the damage in this case. Notice that the random noise is now at a similar level to many of the frequency changes due to damage, shown in Table 1. This, coupled with the limited number of data points (only 8 natural frequencies) leads to severe difficulties for the identification algorithm. In practice the natural frequencies would be estimated using frequency response function measurements in a series of frequency bands. Suppose the standard deviation was 0.1 Hz for the first natural frequency, 0.25 Hz for the next 3 natural frequencies and 1 Hz for the last 4 natural frequencies. Figure 4 shows the mean and standard deviation of the resulting angles, calculated using equation (9) based on 10,000 samples, and shows that the correct parameters would be identified.

7.3. The Standard Method - With Systematic Errors

This example demonstrates the problems associated with systematic errors, introduced in this case from three sources; discretisation errors, a discrete mass not at a node of the finite element model used for the identification and a stiffness reduction covering only half of one element. The 'measurements' are taken from a beam with 40 elements, giving a maximum discretisation error in the first 8 natural frequencies of only 0.1%. Again the changes in the first 8 natural frequencies are 'measured'. Damage is simulated by reducing the stiffness of element number 8 in the fine mesh model by 30%, which is equivalent to changing the stiffness of element 4 in the coarse mesh model. Of course, the damage only covers one half of the element in the coarse mesh model. A mass of 0.1 kg is also added to node 23 of the fine mesh model, corresponding to a point in the middle of element 12 in the coarse mesh model. The iteration strategy outlined in Section 5.1 is used. Subsets of 1, 2 and 3 parameters are tested and the 2 best parameters are retained at each stage. Table 4 shows the resulting parameter selections. Clearly the method has more difficulty locating the errors, and an erroneous parameter $k_1$ is chosen initially. The values of the penalty function $J_m$ (both for the linearised equations and also on convergence) and the
ratio $R_u$ for the second and third parameter selections clearly show the initial choice of $k_1$ was not optimal. Thus the correct regions of damage in the structure may be identified, together with those regions arising from the symmetry in this example.

7.4. Optimising Already Chosen Parameters - With Systematic Errors

What happens if the previously chosen parameters are optimised before the sensitivity matrix is computed and thus before the parameters are selected? Table 5 shows the parameter selected by this iteration strategy which was described in Section 5.2. Only the residuals on convergence are shown. The effect on the choice of the second parameter is small, although significant differences in the choice of the parameters at the third stage are evident. In particular sets of 3 parameters may be chosen that have considerably smaller residuals on convergence than those shown in Table 4. Even so the parameter subsets chosen all correspond to the regions of actual damage, or to those regions arising from the symmetry in this example. Thus the damage is located satisfactorily.

7.5. Multiple Measurement Sets - With Systematic Errors

Suppose that the beam is monitored initially in the undamaged state, and then in 4 damaged states. We also assume that the environment causes the density to change. This situation often arises in concrete structures due to changes in humidity. The measurements are taken from a 40 element beam model, whereas a 20 element beam model will be used to try to locate the errors. The values of the stiffness of element 8 in the fine mesh model and the material density, for each data set, are shown in Table 6. Parameter selection is based on the initial finite element model, and the effects of iteration are neglected. Thus the iteration strategy given in Section 5.1 is used. Indeed accounting for iteration is difficult in this case, because the parameters estimated for each data set would be different, which would then require slightly different sensitivity matrices. Not having a common sensitivity matrix would make the choice of subsequent parameters difficult. Subsets of 1, 2 and 3 parameters are tested and the 4 best parameters are retained at each stage, producing 4 subsets of 1 parameter, 16 subsets of 2 parameters and 64 subsets of 3 parameters. Because of the huge number of subsets, Table 7 only shows the 2 best subsets of 1, 2 and 3 parameters. The results highlight the need to retain more than one or two parameters at each stage since the best subsets containing 2 and 3 parameters would be missed if only the best 2 parameters were chosen at each stage. Note that the density is not one of our candidate parameters, so that Young's Modulus is identified since, like the density, it is a global parameter. Once again the parameter
subsets are reasonably good, although the lack of spatial information and the symmetry within the structure still cause problems.

8. CONCLUSIONS

This paper has summarised a best subspace approach to locate errors in a finite element model of a structure. Sub-optimal methods have been presented to choose the parameter subset that is best able to model the change in a structure due to damage. Strategies to cope with the non-linear penalty function and multiple data sets in typical damage location applications have been discussed. The simulated results are encouraging, although the lack of spatial information and the symmetry of the structure cause considerable problems for these methods. The methods seem reasonably robust to random noise and systematic errors, although large errors, either random or systematic, are likely to cause the selection of incorrect parameters. Although the actual parameters were not always chosen in the example, the regions of damage were correctly identified. In practical applications the proposed approach could be used to identify those areas where more detailed local examination should be concentrated.

ACKNOWLEDGEMENT

Dr. Friswell gratefully acknowledges the support of the Engineering and Physical Sciences Research Council through the award of an Advanced Fellowship.

REFERENCES


Table 1. Natural Frequencies of the Undamaged and Damaged Beam

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Undamaged Frequency (Hz)</th>
<th>Damaged Frequency (Hz)</th>
<th>Percentage Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.96</td>
<td>20.45</td>
<td>2.39</td>
</tr>
<tr>
<td>2</td>
<td>131.3</td>
<td>131.1</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>367.7</td>
<td>366.6</td>
<td>0.31</td>
</tr>
<tr>
<td>4</td>
<td>720.6</td>
<td>711.3</td>
<td>1.29</td>
</tr>
<tr>
<td>5</td>
<td>1191</td>
<td>1172</td>
<td>1.61</td>
</tr>
<tr>
<td>6</td>
<td>1780</td>
<td>1762</td>
<td>1.02</td>
</tr>
<tr>
<td>7</td>
<td>2487</td>
<td>2479</td>
<td>0.32</td>
</tr>
<tr>
<td>8</td>
<td>3313</td>
<td>3303</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table 2. Residuals and Parameter Estimates on Convergence

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$J_m \times 10^6$ First Iteration</th>
<th>$J_m \times 10^6$ On Convergence</th>
<th>Parameter Value On Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(EI)_4$</td>
<td>1.460</td>
<td>0.000000</td>
<td>-30.0 %</td>
</tr>
<tr>
<td>$(\rho A)_{17}$</td>
<td>1.461</td>
<td>0.000028</td>
<td>42.9 %</td>
</tr>
<tr>
<td>$m_{17}$</td>
<td>112.8</td>
<td>94.19</td>
<td>0.183 kg</td>
</tr>
<tr>
<td>$m_{16}$</td>
<td>135.6</td>
<td>139.2</td>
<td>0.197 kg</td>
</tr>
</tbody>
</table>
Table 3. The Selection of Three Parameters - Beam Example with No Errors. The Parameters are Specified by Type and Element/Node number. Thus \((\rho A)_{17}\) is the Mass/Unit Length of Element 17, \((EI)_4\) is the Stiffness of Element 4, \(k_1\) is a Discrete Spring at Node 1 and \(m_{12}\) is a Discrete Mass at Node 12.

<table>
<thead>
<tr>
<th>Choice of Parameter</th>
<th>Parameter 1</th>
<th></th>
<th>Parameter 2</th>
<th></th>
<th>Parameter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(J_m) (\times 10^6)</td>
<td>(J_m) (\times 10^6)</td>
<td>(m)</td>
<td>(R_a) (\times 10^6)</td>
<td>(J_m) (\times 10^6)</td>
</tr>
<tr>
<td>((\rho A)_{17})</td>
<td>154.6</td>
<td>160.4</td>
<td>(m_{12})</td>
<td>1.49</td>
<td>514</td>
</tr>
<tr>
<td>((\rho A)_{12})</td>
<td>8.69</td>
<td>0.007</td>
<td>(m_{12})</td>
<td>1.48</td>
<td>0.026</td>
</tr>
<tr>
<td>((EI)_4)</td>
<td>154.7</td>
<td>160.5</td>
<td>(m_{12})</td>
<td>1.49</td>
<td>515</td>
</tr>
<tr>
<td>((\rho A)_{12})</td>
<td>8.69</td>
<td>0.007</td>
<td>(m_{12})</td>
<td>1.20</td>
<td>24.9</td>
</tr>
<tr>
<td>((\rho A)_{12})</td>
<td>8.69</td>
<td>0.007</td>
<td>(m_{12})</td>
<td>8.69</td>
<td>0.007</td>
</tr>
</tbody>
</table>
Table 4. The Selection of Three Parameters - Beam Example with Systematic Errors. Parameters as Specified in Table 3.

<table>
<thead>
<tr>
<th>Choice of Parameter</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
<th>Parameter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$J_m$ ($\times 10^6$)</td>
<td>$J_m$ ($\times 10^6$)</td>
<td>$J_m$ ($\times 10^6$)</td>
</tr>
<tr>
<td></td>
<td>Conv-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k_1$</td>
<td>140.4</td>
<td>145.5</td>
<td>145.5</td>
</tr>
<tr>
<td></td>
<td>$J_m$</td>
<td>$R_a$</td>
<td>$J_m$</td>
</tr>
<tr>
<td></td>
<td>Conv-</td>
<td>Conv-</td>
<td>Conv-</td>
</tr>
<tr>
<td>$(\rho A)_{17}$</td>
<td>75.8</td>
<td>4.26</td>
<td>77.7</td>
</tr>
<tr>
<td>$(EI)_9$</td>
<td>8.27</td>
<td>32.7</td>
<td>7.29</td>
</tr>
<tr>
<td>$(\rho A)_{12}$</td>
<td>8.27</td>
<td>32.7</td>
<td>7.95</td>
</tr>
<tr>
<td>$(EI)_4$</td>
<td>75.8</td>
<td>4.26</td>
<td>77.5</td>
</tr>
<tr>
<td>$(EI)_9$</td>
<td>8.27</td>
<td>32.7</td>
<td>7.96</td>
</tr>
<tr>
<td>$(\rho A)_{12}$</td>
<td>8.27</td>
<td>32.7</td>
<td>7.24</td>
</tr>
<tr>
<td>$(\rho A)_{17}$</td>
<td>8.35</td>
<td>87.3</td>
<td>7.29</td>
</tr>
<tr>
<td>$(EI)_{12}$</td>
<td>8.33</td>
<td>0.01</td>
<td>-</td>
</tr>
<tr>
<td>$(EI)_9$</td>
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<tr>
<td>$(\rho A)_{12}$</td>
<td>8.30</td>
<td>0.03</td>
<td>7.19</td>
</tr>
<tr>
<td>$(EI)_{12}$</td>
<td>8.30</td>
<td>0.03</td>
<td>7.82</td>
</tr>
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</table>
Table 5. The Selection of Three Parameters - Beam Example with Systematic Errors. Selection of Parameters 2 and 3 Based on Model with Updated Previously Selected Parameters. The Values of the Residuals are on Convergence. Parameters as Specified in Table 3.

<table>
<thead>
<tr>
<th>Choice of Parameter</th>
<th>( J_m ) ( \times 10^6 )</th>
<th>( J_m ) ( \times 10^6 )</th>
<th>( R_a ) ( \times 10^6 )</th>
<th>( J_m ) ( \times 10^6 )</th>
<th>( R_a ) ( \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_1 ) 145.5</td>
<td>( (EI)_4 ) 77.5 4.39</td>
<td>( (\rho A)_{12} ) 7.24 38.8</td>
<td>( (EI)_{12} ) 7.96 34.9</td>
<td>( (EI)_9 ) 7.29 38.6</td>
<td>( (\rho A)_{12} ) 7.95 35.1</td>
</tr>
<tr>
<td>( (\rho A)_{17} ) 154.6</td>
<td>( (EI)_9 ) 7.29 101</td>
<td>( m_{17} ) 1.24 19.6</td>
<td>( m_{17} ) 1.81 1.81</td>
<td>( m_{16} ) 3.19 5.15</td>
<td>( m_{16} ) 3.72 3.72</td>
</tr>
</tbody>
</table>
Table 6. The Parameter Values for the 4 Data Sets

<table>
<thead>
<tr>
<th>Data Set Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in $(EI)_8$</td>
<td>-1%</td>
<td>-3%</td>
<td>-5%</td>
<td>-10%</td>
</tr>
<tr>
<td>Change in $\rho$</td>
<td>+2%</td>
<td>-3%</td>
<td>-1%</td>
<td>+3%</td>
</tr>
</tbody>
</table>

Table 7. Parameter Selection for the Beam Example with 4 Data Sets Containing Systematic Errors. 84 Subsets of Parameters were Generated, but Only the Best 2 Selections for 1, 2 and 3 Parameters are Shown. Parameters as Specified in Table 3.

<table>
<thead>
<tr>
<th>Choice of Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter 1</td>
</tr>
<tr>
<td>Angle</td>
</tr>
<tr>
<td>$(E)_1$</td>
</tr>
<tr>
<td>$m_{20}$</td>
</tr>
<tr>
<td>$(\rho A)_{17}$</td>
</tr>
<tr>
<td>$(EI)_4$</td>
</tr>
<tr>
<td>$(EI)_4$</td>
</tr>
<tr>
<td>$(\rho A)_{17}$</td>
</tr>
<tr>
<td>$E$</td>
</tr>
<tr>
<td>$E$</td>
</tr>
<tr>
<td>$E$</td>
</tr>
<tr>
<td>$m_{16}$</td>
</tr>
<tr>
<td>$m_{16}$</td>
</tr>
</tbody>
</table>
Figure 1. Sub-Space Angles for a 30% Change to the Stiffness of Element 4
Figure 2. Mean Sub-Space Angles for a 30% Change to the Stiffness of Element 4. 10000 Samples with 0.1% Measurement Noise, Error Bars Denote Standard Deviations
Figure 3. Mean Sub-Space Angles for a 30% Change to the Stiffness of Element 4.
0.5% Measurement Noise, Error Bars Denote Standard Deviations
Figure 4. Mean Sub-Space Angles for a 30% Change to the Stiffness of Element 4. Measurement Noise Standard Deviation 0.1 Hz for Mode 1, 0.25 Hz for Modes 2-4, 1 Hz for Modes 5-8, Error Bars Denote Standard Deviations.