Static and free vibration analysis of functionally graded carbon nanotube reinforced skew plates

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ABSTRACT

The remarkable mechanical and sensing properties of carbon nanotubes (CNTs) suggest that they are ideal candidates for high performance and self-sensing composites. However, the study of CNT-based composites is still under development. This paper provides results of static and dynamic numerical simulations of thin and moderately thick functionally graded (FG-CNTRC) skew plates with uniaxially aligned reinforcements. The shell element is formulated in oblique coordinates and based on the first-order shear deformation plate theory. The theoretical development rests upon the Hu–Washizu principle. Independent approximations of displacements (bilinear), strains and stresses (piecewise constant subregions) provide a consistent mechanism to formulate an efficient four-noded skew element with a total of twenty degrees of freedom. An invariant definition of the elastic transversely isotropic tensor is employed based on the representation theorem. The FG-CNTRC skew plates are studied for a uniform and three different distributions (two symmetric and one asymmetric) of CNTs. Detailed parametric studies have been carried out to investigate the influences of skew angle, CNT volume fraction, thickness-to-width ratio, aspect ratio and boundary conditions. In addition, the effects of fiber orientation are also examined. The obtained results are compared to the FE commercial package ANSYS and the limited existing bibliography with good agreement.

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1. Introduction

Since the discovery of carbon nanotubes (CNTs) by Ijima [1] in 1991, many researchers have investigated their unique capabilities as reinforcements in composite materials. Due to their remarkable mechanical, electrical and thermal properties, carbon nanotubes are considered ideal reinforcing fibers for advanced high strength materials and smart materials with self sensing capabilities [2,3]. In actual structural applications, it is important to develop theoretical models in order to predict the response of structural elements made of carbon nanotube-reinforced composites (CNTRC). In particular, skew plates are widely employed in civil and aeronautical engineering applications such as panels in skew bridges, construction of wings, tails and fins of swept-wing aircraft, etc. However, due to the mathematical difficulties involved in their formulation, works on the static and dynamic analysis of CNTRC skew elements are scarce in the literature [4].

The number of publications dealing with static and dynamic analysis of CNTRC structural elements have increased considerably in recent years. Wuite and Adali [5] studied the bending behavior of classical symmetric cross-ply and angle-ply laminated beams reinforced by aligned CNTs and isotropic beams reinforced by randomly oriented CNTs. By using a micromechanical constitutive model based on the Mori–Tanaka method, they highlighted that small percentages of CNT reinforcement lead to significant improvement in beam stiffness. Vodenitcharova and Zhang [6] developed a continuum model for the uniform bending and bending-induced buckling of a straight nanocomposite beam with circular cross section reinforced by a single-walled carbon nanotube (SWNT). The results showed that although the addition of a matrix to a SWNT increases the load carrying capacity, the thicker matrix layers the SWNT buckles locally at smaller bending angles and greater flattening ratios. Formica et al. [7] studied the vibrational properties of cantilevered CNTRC plates with an
Eshelby–Mori–Tanaka approach and finite element modeling. The results demonstrated the ability of CNTs to tune the vibrational properties of composites and increase the fundamental frequencies up to 500%. These exceptional properties have motivated many researchers to optimize the contribution of CNTs. According to this principle, Arani et al. [8] investigated analytically and numerically the buckling behavior of CNTRC rectangular plates. Based on classical laminate plate theory and the third-order shear deformation theory for moderately thick plates, they optimized the orientation of CNTs to achieve the highest critical load. Another example of this interest is the research carried out by Rokni et al. [9]. By dividing a beam along its longitudinal and thickness direction with the inclusion proportion as the design variable, they proposed a new two-dimensional optimum distribution of reinforcements of a polymer composite micro-beams to maximize the fundamental natural frequency given a weight percentage of CNTs.

Functionally graded materials (FGMs) belong to a branch of advanced materials characterized by spatially varying properties. This concept has promoted the development of a wide range of applications of functionally graded composite materials since its origin in 1984 (see e.g. [10]). Inspired by this idea, Shen [11] proposed non-uniform distributions of CNTs within an isotropic matrix. In this work, nonlinear vibration of functionally graded CNT-reinforced composite (FG-CNTRC) plates in thermal environments was presented. Researchers have employed many different methodologies to model FG-CNTRCs and most of them are recorded in a recent review by Liew et al. [12]. Zhu et al. [13] carried out bending and free vibration analysis of FG-CNTRC plates by using a finite element model based on the first-order shear deformation plate theory (FSDT). Ke et al. [14] presented nonlinear free vibration analysis of FG-CNTRC beams within the framework of Timoshenko beam theory and Ritz method solved by a direct iterative technique. They concluded that symmetrical distributions of CNTs provide higher linear and nonlinear natural frequencies for FG-CNTRC beams than with uniform or unsymmetrical distribution of CNTs. Zhang et al. [15] proposed a meshless local Petrov–Galerkin approach based on the moving Kriging interpolation technique to analyze the geometrically nonlinear thermoelastic behavior of functionally graded plates in thermal environments. Shen and Zhang [16] analyzed the thermal buckling and postbuckling behavior of uniform and symmetric FG-CNTRC plates under in-plane temperature variation. These results showed that the buckling temperature as well as thermal postbuckling strength of the plate can be increased with functionally graded reinforcement. However, in some cases the plate with intermediate nanotube volume fraction may not present intermediate buckling temperature and initial thermal postbuckling strength. Aragh et al. [17] proposed an Eshelby–Mori–Tanaka approach and a 2-D generalized differential quadrature method (GDQM) to investigate the vibrational behavior of rectangular plates resting on elastic foundations. Yas and Heshmati used Timoshenko beam theory to analyze the vibration of straight uniform [18] and non-uniform [19] FG-CNTRC beams subjected to moving loads. Alibeigloo and Liew [20] studied the bending behavior of FG-CNTRC plates with simply supported edges subjected to thermo-mechanical loading conditions by three-dimensional elasticity theory and using the Fourier series expansion and state-space method. This work was extended by Alibeigloo and Emtehani [21] for various boundary conditions by using the differential quadrature method. Zhang et al. [22] proposed a state-space Levy method for the vibration analysis of FG-CNT composite plates subjected to in-plane loads based on higher-order shear deformation theory. This research analyzed three different symmetric distributions of the reinforcements along the thickness, namely UD, FG-X and FG-O. They concluded that FG-X provides the largest frequency and critical buckling in-plane load. Whereas, the frequency for the FGO-CNT plate was the lowest. Wu and Li [23] used a unified formulation of Reissner’s mixed variational theorem (RMVT) based finite prism methods (FPMs) to study the three-dimensional free vibration behavior of FG-CNTRC plates. Free vibration analyses of quadrilateral laminated plates were carried out by Malekzadeh and Zarei [24] using first shear deformation theory and discretization of the spatial derivatives by the differential quadrature method (DQM). Furthermore, mesh-free methods, employed in many different fields such as elastodynamic problems [25] and wave equations [26], have also been widely employed in the simulation of FG-CNTRCs. Zhang et al. [27] employed a local Kriging meshless method to evaluate the mechanical and thermal buckling behaviors of ceramic–metal functionally graded plates (FGPs). Lei et al. [28] presented parametric studies of the dynamic stability of CNTRC-FG cylindrical panels under static and periodic axial force using the mesh-free-kr–Ritz method and the Eshelby–Mori–Tanaka homogenization framework. Lei et al. [29] employed this methodology to carry out vibration analysis of thin-to-moderately thick laminated FG-CNT rectangular plates. Zhang and Liew [30] presented detailed parametric studies of the large deflection behaviors of quadrilateral FG-CNT for different types of CNT distributions. They showed that the geometric parameters such as side angle, thickness-to-width ratio or plate aspect ratios are more significant than material parameters such as CNT distribution and CNT volume fraction. Zhang et al. [31] employed the IMLS-Ritz method to assess the postbuckling behavior of FG-CNT plates with edges elastically restrained against translation and rotation under axial compression. Some other results can be found in the literature dealing with the buckling analysis of FG-CNTRC thick plates resting on Winkler [32] and Pasternak foundations [33], free vibration analysis of triangular plates [34], cylindrical panels [35,36], three-dimensional free vibration analysis of FG-CNTRC plates [37], vibration of thick functionally graded carbon nanotube-reinforced composite plates resting on elastic Winkler foundations [38], vibration analysis of functionally graded carbon nanotube reinforced thick plates with elastically restrained edges [30], etc.

In the case of skew plates, the verification of their mathematical formulation is difficult because of the lack of exact solutions, and those available in literature are based on approximate methods. Over the past four decades, a lot of research has been carried out on the study of isotropic skew plates [39–42]. In contrast, research work dealing with the analysis of anisotropic skew plates is rather scant, and even more so for FG-CNT composite materials. However, Zhang et al. [30] obtained the buckling solution of FG-CNT reinforced composite moderately thick skew plates using the element-free IMLS-Ritz method and first-order shear deformation theory (FSDT). The same authors [4] also provided approximate solutions for the free vibration of uniform and a symmetric distribution of the volume fraction of CNT in moderately thick FG-CNT skew plates. This methodology was also employed by Lei et al. [43] to perform buckling analysis of thick FG-CNT skew plates resting on Pasternak foundations. Geometrically nonlinear large deformation analysis of FG-CNT skew plates resting on Pasternak foundations was carried out by Zhang and Liew [44].

In this paper, we develop an efficient finite element formulation based on the Hu–Washizu principle to obtain approximate solutions for static and free vibration of various types of FG-CNTRC skew plates with moderate thickness. The shell theory is formulated in oblique coordinates and includes the effects of transverse shear strains by first-order shear deformation theory (FSDT). An invariant definition of the elastic transversely isotropic tensor based on the representation theorem is also defined in oblique coordinates. Independent approximations of displacements, strains and stresses (piecewise constant within subregions) provide a consistent mechanism to formulate four-noded skew elements with a total of twenty degrees of freedom. A set of eigenvalue equations for the FG-CNTRC skew plate vibration is
derived, from which the natural frequencies and mode shapes can be obtained. Detailed parametric studies have been carried out to investigate the influences of skew angle, carbon nanotube volume fraction, plate thickness-to-width ratio, plate aspect ratio, boundary conditions and the distribution profile of reinforcements (uniform and three non-uniform distributions) on the static and dynamic response of the FG-CNTRC skew plates. The results are compared to commercial codes and the limited existing bibliography with very good agreement.

2. Functionally graded CNTRC plates

Fig. 1 shows the four types of FG-CNTRC skew plates considered in this paper, with length $a$, width $b$, thickness $t$ and fiber orientation angle $\varphi$. UD-CNTRC represents the uniform distribution and FG-V, FG-O and FG-X CNTRC are the functionally graded distributions of carbon nanotubes in the thickness direction of the composite skew plates. The effective material properties of the two-phase nanocomposites mixture of uniaxially aligned CNTs reinforced by a polymeric matrix, can be estimated according to the Mori–Tanaka scheme [45] or the rule of mixtures [3,46]. The accuracy of the extended rule of mixtures (EROM) has been widely discussed and a remarkable synergism with the Mori–Tanaka scheme for functionally graded ceramic–metal beams is reported in [17]. Due to the simplicity and convenience, in the present study, the extended rule of mixture was employed by introducing the CNT efficiency parameters and the effective material properties of CNTRC skew plates can thus be written as [11]

$$E_{11} = \eta_1 V_{\text{CNT}} E_{11}^{\text{CNT}} + V_m E^m \quad (1a)$$
$$\eta_2 = \frac{V_{\text{CNT}}}{E_{22}^{\text{CNT}}} + \frac{V_m}{E^m} \quad (1b)$$
$$\eta_3 = \frac{V_{\text{CNT}}}{G_{12}^{\text{CNT}}} + \frac{V_m}{G^m} \quad (1c)$$

where $E_{11}^{\text{CNT}}$ and $G_{12}^{\text{CNT}}$ indicate the Young’s moduli and shear modulus of SWCNTs, respectively, and $E^m$ and $G^m$ represent corresponding properties of the isotropic matrix. To account for the scale-dependent material properties, the CNT efficiency parameters, $\eta_j (j = 1,2,3)$, were introduced and can be calculated by matching the effective properties of the CNTRC obtained from a molecular dynamics (MD) or multi-scale simulations with those from the rule of mixtures. $V_{\text{CNT}}$ and $V_m$ are the volume fractions of the carbon nanotubes and matrix, respectively, and the sum of the volume fractions of the two constituents should equal unity. Similarly, the thermal expansion coefficients, $a_{11}$ and $a_{22}$, in the longitudinal and transverse directions respectively, Poisson’s ratio $\nu_{12}$ and the density $\rho$ of the nanocomposite plates can be determined in the same way as

$$v_{12} = V_{\text{CNT}} v_{12}^{\text{CNT}} + V_m v^m \quad (2a)$$
$$\rho = V_{\text{CNT}} \rho^{\text{CNT}} + V_m \rho^m \quad (2b)$$
$$\nu_{11} = \frac{V_{\text{CNT}} \nu_{11}^{\text{CNT}} + V_m \nu^m}{1 + \nu^m} \quad (2c)$$
$$\nu_{22} = \frac{(1 + \nu_{12}^m) V_{\text{CNT}} \nu_{22}^{\text{CNT}} + (1 + \nu^m) V_m \nu^m - v_{12} \nu_{11}}{1 - 2 \nu^m} \quad (2d)$$

where $v_{12}^{\text{m}}$ and $v^m$ are Poisson’s ratios, and $\nu_{11}^{\text{CNT}}$, $\nu_{22}^{\text{CNT}}$ and $\nu^m$ are the thermal expansion coefficients of the CNT and matrix, respectively. Note that $v_{12}$ is considered as constant over the thickness of the functionally graded CNTRC skew plates.

And the other effective mechanical properties are

$$E_{33} = E_{22}, \quad G_{13} = G_{12}, \quad G_{23} = \frac{1}{2} \left( \frac{E_{22}}{1 + \nu_{23}} + \frac{E_{33}}{1 + \nu_{33}} \right)$$

$$v_{13} = v_{12}, \quad v_{31} = v_{21}, \quad v_{32} = v_{23} = v_{21}, \quad (3)$$
$$v_{21} = v_{22} / E_{11}$$

The uniform and three types of functionally graded distributions of the carbon nanotubes along the thickness direction of the nanocomposite skew plates shown in Fig. 1 are assumed to be

$$V_{\text{CNT}} = V_{\text{CNT}}^0 \quad (\text{UD CNTRC})$$
$$V_{\text{CNT}} = \frac{t_2}{t} V_{\text{CNT}}^0 \quad (\text{FG-X CNTRC})$$
$$V_{\text{CNT}} = (1 + \frac{2t_2}{t}) V_{\text{CNT}}^0 \quad (\text{FG-V CNTRC})$$
$$V_{\text{CNT}} = 2 \left( 1 - \frac{2t_2}{t} \right) V_{\text{CNT}}^0 \quad (\text{FG-O CNTRC}) \quad (4)$$

3. Finite element formulation

3.1. Parametrization of the geometry

Consider CNTRC skew plate of length $a$, width $b$, thickness $t$ and skew angle $\alpha$ as shown in Fig. 1. The midsurface of the shell to be considered in this paper is given in terms of skew coordinates ($\theta^1, \theta^2$), hence the change of coordinates is given by

$$x = \theta^1 + \theta^2 \cos \alpha$$
$$y = \theta^2 \sin \alpha$$
$$z = \theta^3 \quad (5)$$

This parametrization leads a covariant basis $\tilde{a}$, defined by Eq. (6)

$$\tilde{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{a}_2 = \begin{bmatrix} \cos \alpha \\ \sin \alpha \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(6)

The covariant metric tensor is noted by $a$ has a value of $\sin^2 \alpha$ and leads a contravariant basis $a^i$ defined by Eq. (7)

$$a^1 = \begin{bmatrix} 1 \\ -\tan^{-1} \alpha \\ 0 \end{bmatrix}, \quad a^2 = \begin{bmatrix} \csc \alpha \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad a^3 = \tilde{a}_3$$

(7)
3.2. Variational formulation, displacement field, stresses and strains of CNTRC skew plates

The theoretical formulation is derived by a variational formulation. Denoting by $\mathcal{U}(\gamma)$ the strain energy and by $\gamma$ and $\sigma$ the vectors containing the strain and stress components, respectively, a modified potential of Hu–Washizu assumes the form [47]

$$
\Pi_{\text{int}}[\mathbf{v},\gamma;\sigma] = \int_0^1\left[\mathcal{U}(\gamma) - \sigma^T (\gamma - \mathbf{Dv})\right] - \int_{S_k} (\mathbf{v} - \mathbf{v}) \sigma n dS - \int_{S_k} \Pi_0 dS
$$

(8)

In Eq. (8), $\mathbf{v}$ and the index $b$ represent the displacement vector and the body forces, respectively, whereas $\gamma$ are prescribed displacements on the part of the boundary in which displacements are prescribed ($S_b$).

The displacement field is constructed by first-order shear deformation. Hence the in-plane deformation $\gamma_{xy}$ is expressed in terms of the extensional ($a_{1,xy}$) and flexural ($a_{1,xy}$) components of the Cauchy–Green strain tensor as

$$
\gamma_{xy} = a_{1,xy} + \theta^* \gamma_{xy}^{\text{a}}.
$$

(9)

Denoting by $V_1$ and $V_3$ the tangential displacements of the mid-surface in the $\theta^*$ and $\theta^\mu$ directions, and by $\phi_{xy}$ the rotations about the $\theta^\nu$ lines, the strains in terms of the aforementioned displacements and rotations have the form

$\gamma_{xy}^{\text{a}} = \theta^* \gamma_{xy}^{\text{a}}.$

(10a)

$\gamma_{xy} = \theta^* \gamma_{xy}^{\text{a}} = \theta^* \gamma_{xy}^{\text{a}}.$

(10b)

$\gamma_{xy}^{\text{a}} = \theta^* \gamma_{xy}^{\text{a}}.$

(10c)

In Eqs. (10) $e_{xy}$ denote the permutation tensor associated with the undeformed surface and a double bar ($\overline{\gamma}$) signifies covariant differentiation with respect to the undeformed surface. In aovrformal

$$
0^{\overline{\gamma}} = \left\{ \begin{array}{c}
0_{1,11}^{\overline{\gamma}} \\
0_{1,22}^{\overline{\gamma}} \\
2 a_{1,12}^{\overline{\gamma}}
\end{array} \right\}, \quad 0^{\gamma} = \left\{ \begin{array}{c}
0_{1,11}^{\gamma} \\
0_{1,22}^{\gamma} \\
2 a_{1,12}^{\gamma}
\end{array} \right\} \quad \text{and} \quad 0^{\gamma} = \left\{ \begin{array}{c}
0_{1,13}^{\gamma} \\
0_{1,23}^{\gamma} \\
0_{2,12}^{\gamma}
\end{array} \right\}
$$

(11)

The thin body assumption is considered in the z-direction, and thus it is often possible to neglect the transverse normal stress $s^{33}$. The stress–strain relationships are defined by

$$
s^{\beta} = \frac{\partial \Phi}{\partial \gamma_{\alpha \beta}} ,
$$

(12)

$$
s^{33} = 2E^{333} \gamma_{13}^{\beta} ,
$$

$$
s^{33} = 0
$$

And the free-energy density takes the form

$$
\Phi = \frac{1}{2} C_{ijkl} s^{ij} \gamma_{kl} + 2E^{333} \gamma_{33} \gamma_{33}
$$

(13)

3.3. Linearly elastic transversely isotropic constitutive matrix in non-orthogonal coordinates

The definition of non-orthogonal coordinates requires a coherent definition of the stress–strain relationships. On the basis of the representation theorems of transversely isotropic tensors developed by Spencer [48], Lumbarda and Chen [49] obtained the constitutive tensor of linear elastic transversely isotropic materials in a general coordinates system as

$$
C_{ijkl} = \left\{ \begin{array}{c}
C_{ijkl}^{\text{a}} \\
C_{ijkl}^{\text{b}} \\
C_{ijkl}^{\text{c}} \\
C_{ijkl}^{\text{d}}
\end{array} \right\}
$$

(14)

$\Gamma^f$ are a set of linearly independent fourth order tensors that form a basis of an algebra of order 6 and the $c_i$ are six elastic parameters. In component form, the fourth-order tensors $I_i$ are defined by

$$
I_{ijkl}^{(a)} = \frac{1}{2} (a^a a^b + a^a a^b)
$$

(15a)

$$
I_{ijkl}^{(b)} = a^a a^b
$$

(15b)

$$
I_{ijkl}^{(c)} = a^a a^b
$$

(15c)

$$
I_{ijkl}^{(d)} = a^a a^b
$$

(15d)

$$
I_{ijkl}^{(e)} = \frac{1}{2} (a^a a^b + a^a a^b + a^a a^b + a^a a^b)
$$

(15e)

$$
I_{ijkl}^{(f)} = a^a a^b
$$

(15f)

where $n$ are the rectangular components of an unit vector parallel to the axis of the transverse isotropy, defined in the mid-plane of the skew plate as $\bar{n} = (\cos \phi, \sin \phi, 0)$ (see Fig. 1), and $a^a$ are the components of the contravariant basis $a^a$ defined in Eq. (7). The material parameters, $c_i$, are defined as

$$
c_1 = 2\mu, \quad c_2 = \lambda, \quad c_3 = c_4 = \alpha, \quad c_5 = 2(\mu_0 - \mu), \quad c_6 = \beta
$$

(16)

The material parameters $c_i$ depend on five elastic constants: $\mu$ and $\lambda$, shear modulus within the plane of isotropy and the Lamé constant, the out-of-plane elastic shear modulus $\mu_0$, $\alpha$ and $\beta$. In matrix notation the 4th order elasticity tensor of transversely isotropic material for a preferred $x$ direction in a Cartesian coordinate system gives

$$
C = \begin{bmatrix}
2\alpha + \beta + \lambda - 2\mu + 4\mu_0 & \alpha + \lambda & \alpha + \lambda & 0 & 0 & 0 \\
\alpha + \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\alpha + \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_0
\end{bmatrix}
$$

(17)

The relation between elastic invariant constants and the engineering constants can be found by comparing Eq. (17) with the classical transversely isotropic stiffness tensor. This comparison leads to

$$
\alpha = \frac{E_{11}(E_{11} - E_{22})E_{22}v_{12}}{(E_{11} + E_{22}v_{12})(E_{11} - E_{22}v_{12}(1 + 2v_{12}))}
$$

(18a)

$$
\lambda = \frac{E_{11}E_{22}v_{12}(1 + v_{12})}{(E_{11} + E_{22}v_{12})(E_{11} - E_{22}v_{12}(1 + 2v_{12}))}
$$

(18b)

$$
\mu = \frac{E_{11} + 2E_{22}}{2E_{11} + 2E_{22}}
$$

(18c)

$$
\mu_0 = G_{12}
$$

(18d)

$$
\beta = \frac{1}{2} \left( -8G_{12} + \frac{E_{11}E_{22}}{E_{11} + E_{22}}v_{12} + \frac{E_{11}(E_{11} + E_{22} - 6E_{22}v_{12})}{E_{11} - E_{22}v_{12}(1 + 2v_{12})} \right)
$$

(18e)

Once the constitutive tensor is obtained, the plane stress stiffness matrix can be obtained numerically by deleting the rows and columns associated with the $z$-direction in the compliance matrix. By inverting the resulting compliance matrix, the constitutive equations are written in Voigt’s notation in the form

$$
\begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{12} & S_{22} & S_{23} \\
S_{13} & S_{23} & S_{33}
\end{bmatrix}
\begin{bmatrix}
Q_{11}(z) & Q_{12}(z) & Q_{13}(z) \\
Q_{12}(z) & Q_{22}(z) & Q_{23}(z) \\
Q_{13}(z) & Q_{23}(z) & Q_{33}(z)
\end{bmatrix}
\begin{bmatrix}
S_{11} \\
S_{12} \\
S_{13}
\end{bmatrix}
$$

(19)
\[
\begin{align*}
&\left(C_i^{\parallel}, C_i^{\perp}, C_i^{\cdot}\right) = \int_{-h/2}^{h/2} Q_{ij} \cdot (1, z, z^2) dz \quad (i, j = 1, 2, 6), \\
&C_i^{\parallel} = \frac{1}{K} \int_{-h/2}^{h/2} Q_{ij} dz \quad (i, j = 4, 5)
\end{align*}
\] (20)

Note that \(Q_{ij}\) varies with \(z\) according to the grading profile of the CNTRC along the thickness. \(K\) denotes the transverse shear correction factor for FGM, given by \([50]\)

\[
ks = \frac{6 - (v_1V_1 + v_3V_3)}{5}
\] (21)

### 3.4. Stiffness matrix of skew plate element

The strain-energy density per unit of area at the reference surface can be defined by

\[
U = \int_{-h/2}^{h/2} \Phi dz
\] (22)

From Eq. (9) and Eq. (13), the strain-energy density can be expressed as

\[
U = \frac{1}{2} \int_{-h/2}^{h/2} \left[ E \varepsilon^{\parallel 2} \varepsilon^{\parallel 2} + \varepsilon^{\perp 2} \varepsilon^{\perp 2} + 2G \varepsilon^{\cdot 2} + 2\varepsilon^{\cdot 2} + 2\varepsilon^{\cdot 2} \right] dz
\] (23)

Expression (23) for the strain energy can be represented as the sum of the extensional (\(U_1\)), bending (\(U_2\)), coupling (\(U_C\)) and transverse shear (\(U_\Sigma\)) strain energy as

\[
U_{\text{total}} = U_k + U_b + U_c + U_s
\]

\[
= \frac{1}{2} \left( A_{11} \gamma_1^2 + A_{12} \gamma_1 \gamma_2 + A_{13} \gamma_1 \gamma_3 + A_{22} \gamma_2^2 + A_{23} \gamma_2 \gamma_3 + A_{33} \gamma_3^2 \right)
\] (24)

### 3.4.1. Discretization

The shell element derived in the present study is a four-noded skewed isoparametric finite element (see Fig. 2) with five degrees of freedom at each node: three physical components of the displacements \(u_i, u_2, u_3\) and two components of the rotations \(\varphi_x, \varphi_y\). Bilinear shape functions \(N_k\) are chosen for the physical components of the displacements and rotations in the following way

\[
u_i = \sum_{k=1}^{4} u_i^k N_k \quad \text{and} \quad \varphi_x = \sum_{k=1}^{4} \varphi_x^k N_k;
\] (25)

\[
N_k = \frac{1}{4} \left( 1 + \xi_k \xi \right) \left( 1 + \eta_k \eta \right), \quad i = 1, 2, 3 \quad \text{and} \quad \alpha = 1, 2.
\] (26)

As mentioned before, the use of the Hu–Washizu principle and the independent approximation of strain and stress yields a series of desirable features important for the reliability, convergence behavior, and efficiency of the elemental formulation such as the avoidance of superfluous energy and zero energy modes. Furthermore, the discrete approximation is drawn in a consistent manner from the general theory of the continuum and the mechanical behavior of the finite element, without resorting to special manipulations or computational procedures. In addition, it has been shown \([47,51,52]\) that essential prerequisites for the achievement of these goals are: the identification of constant and higher-order deformational modes which are contained in the displacement/rotation assumptions, the realization that the constant terms are necessary for convergence, and that higher-order terms reappear in different strain components. Therefore, our approximation does not need to retain the higher-order terms in two different strain components (they are needed only to inhibit a mode).

For instance, the following assumptions for the extensional strains have been shown to serve the aforementioned goals

\[
\begin{align*}
\gamma_{11} &= \frac{\gamma_{11}}{\nu_{11}} + \gamma_{11} \eta, \\
\gamma_{22} &= \frac{\gamma_{22}}{\nu_{22}} + \gamma_{22} \xi, \\
\gamma_{12} &= \frac{\gamma_{12}}{\nu_{12}} + \gamma_{12} \xi \eta.
\end{align*}
\] (27)

Note that, according to the above ideas, the underlined terms in Eqs. (27) are not considered. The elimination of such terms allows the reduction of excessive internal energy and to improve convergence. Furthermore, the replacement of the linear variation of the strains and stresses by piecewise constant approximations leads to computational advantages that are most important in repetitive computations. The piecewise constant approximations can be improved by introducing four subdomains over the finite element (see Fig. 3). For example, Fig. 4 illustrates the piecewise approximation of \(\gamma_{11}\) and \(\gamma_{22}\) over two subdomains. The membrane shear strain \(\gamma_{12}\) is approximated by a constant.

Considering the piecewise approximations through the four subdomains and expressing strains in physical components (\(\varepsilon, \kappa, \gamma\)), the extensional, bending and shear strain over every subdomain are defined as

\[
\begin{align*}
\varepsilon_{11} &= \left\{ \begin{array}{ll}
\varepsilon_{11}^A & \text{in } A_1 + A_2, \\
\varepsilon_{11}^B & \text{in } A_{11} + A_{12}, \\
\varepsilon_{11}^C & \text{in } A_{12} + A_{13}, \\
\varepsilon_{11}^D & \text{in } A_{13} + A_{14}, \\
\varepsilon_{11}^E & \text{in } A_{14} + A_{15}.
\end{array} \right.
\end{align*}
\] (28)
Bending strains \( (\kappa_{11}, \kappa_{22}, \kappa_{12}) \)

\[
\kappa_{11} = \begin{cases} \\
\kappa_{11}^{0} & \text{in } A_{1} + A_{II} \\
\kappa_{11}^{0} & \text{in } A_{II} + A_{IV},
\end{cases}
\]

\[
\kappa_{22} = \begin{cases} \\
\kappa_{22}^{0} & \text{in } A_{1} + A_{IV} \\
\kappa_{22}^{0} & \text{in } A_{III} + A_{IV} \text{ and } \kappa_{12} = \kappa_{12} \text{ in } A
\end{cases}
\]  

(29)

Shear strains \( (\gamma_{13}, \gamma_{23}) \)

\[
\gamma_{1} = \begin{cases} \\
\gamma_{1}^{0} & \text{in } A_{1} + A_{III} \\
\gamma_{1}^{0} & \text{in } A_{III} + A_{IV}
\end{cases}
\]  

\[
\gamma_{2} = \begin{cases} \\
\gamma_{2}^{0} & \text{in } A_{1} + A_{III} \\
\gamma_{2}^{0} & \text{in } A_{III} + A_{IV}
\end{cases}
\]  

(30)

As a consequence of this approximation, the strain energy term in the Hu–Washizu variational principle takes the form of

\[
\int_{A} U \, dA = \int_{A_{1}} U_{1} \, dA + \ldots + \int_{A_{IV}} U_{IV} \, dA = \sum_{i=1}^{IV} \int_{A_{i}} U_{i} \, dA
\]

\[
= \frac{1}{2} \varepsilon^{T} D_{E} \varepsilon + \frac{1}{2} \kappa^{T} D_{B} \kappa + \frac{1}{2} \varepsilon^{T} D_{C} \varepsilon + \frac{1}{2} \kappa^{T} D_{C} \kappa + \frac{1}{2} \gamma^{T} D_{S} \gamma
\]

(31)

where the vectors \( \varepsilon, \kappa, \) and \( \gamma \) are defined by

\[
\varepsilon = \begin{bmatrix} \varepsilon_{11}^{A} \\
\varepsilon_{22}^{A} \\
\varepsilon_{12}^{A}
\end{bmatrix}, \quad \kappa = \begin{bmatrix} \kappa_{11}^{A} \\
\kappa_{22}^{A} \\
\kappa_{12}^{A}
\end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_{11}^{A} \\
\gamma_{22}^{A} \\
\gamma_{12}^{A}
\end{bmatrix}
\]

(32)

The matrices \( D_{E}, D_{B}, D_{C}, \) and \( D_{S} \) are the discretized elasticity matrices that depend on the geometry of the surface—i.e., on the contravariant \((a^{ij})\) and covariant \((a_{ij})\) components of the metric tensors—and can be represented as follows

\[
D_{E} = \begin{bmatrix} \\
J_{11} a_{11} D_{1} (1,1) \, dA & J_{11} a_{12} D_{1} (1,2) \, dA & \ldots & J_{11} a_{1N} D_{1} (1,N) \, dA \\
J_{12} a_{11} D_{1} (2,1) \, dA & J_{12} a_{12} D_{1} (2,2) \, dA & \ldots & J_{12} a_{1N} D_{1} (2,N) \, dA \\
\vdots & \vdots & \ddots & \vdots \\
J_{N1} a_{11} D_{1} (N,1) \, dA & J_{N1} a_{12} D_{1} (N,2) \, dA & \ldots & J_{N1} a_{1N} D_{1} (N,N) \, dA
\end{bmatrix}_{\text{sym}}
\]

(33)

\[
D_{B} = \begin{bmatrix} \\
J_{11} a_{11} D_{2} (1,1) \, dA & J_{11} a_{12} D_{2} (1,2) \, dA & \ldots & J_{11} a_{1N} D_{2} (1,N) \, dA \\
J_{12} a_{11} D_{2} (2,1) \, dA & J_{12} a_{12} D_{2} (2,2) \, dA & \ldots & J_{12} a_{1N} D_{2} (2,N) \, dA \\
\vdots & \vdots & \ddots & \vdots \\
J_{N1} a_{11} D_{2} (N,1) \, dA & J_{N1} a_{12} D_{2} (N,2) \, dA & \ldots & J_{N1} a_{1N} D_{2} (N,N) \, dA
\end{bmatrix}_{\text{sym}}
\]

(34)

Furthermore, the parameters for the stress resultants are expressed by the following vector forms

\[
N^T = \begin{bmatrix} N_{11}^{A} & N_{11}^{B} & N_{12}^{C} & N_{12}^{D} & N_{12}^{E} \end{bmatrix},
\]

\[
M^{T} = \begin{bmatrix} M_{11}^{A} & M_{11}^{B} & M_{12}^{C} & M_{12}^{D} & M_{12}^{E} \end{bmatrix} \text{ and }
\]

\[
Q^{T} = \begin{bmatrix} Q_{11}^{A} & Q_{11}^{B} & Q_{12}^{C} & Q_{12}^{D} & Q_{12}^{E} \end{bmatrix}.
\]

(35)

In addition, by introducing the matrices

\[
A_{\kappa} = A_{\kappa} = \begin{bmatrix} A_{1} + A_{II} & 0 & 0 & 0 & 0 \\
0 & A_{III} + A_{IV} & 0 & 0 & 0 \\
0 & 0 & A_{1} + A_{IV} & 0 & 0 \\
0 & 0 & 0 & A_{II} + A_{IV} & 0 \\
0 & 0 & 0 & 0 & A
\end{bmatrix}
\]

(36a)

\[
A_{Q} = \begin{bmatrix} A_{1} + A_{II} & 0 & 0 & 0 & 0 \\
0 & A_{III} + A_{IV} & 0 & 0 & 0 \\
0 & 0 & A_{1} + A_{IV} & 0 & 0 \\
0 & 0 & 0 & A_{II} + A_{IV} & 0 \\
0 & 0 & 0 & 0 & A
\end{bmatrix}
\]

(36b)

along with the discretized strain-displacement relationships, the bilinear approximations for the displacements and rotations, and also the discrete parameters for the strains and stresses, the discrete form of the generalized variational principle of Hu–Washizu is given by

\[
\Pi_{\text{HW}} = \frac{1}{2} \varepsilon^{T} D_{E} \varepsilon + \frac{1}{2} \kappa^{T} D_{B} \kappa + \frac{1}{2} \varepsilon^{T} D_{C} \varepsilon + \frac{1}{2} \kappa^{T} D_{C} \kappa + \frac{1}{2} \gamma^{T} D_{S} \gamma
\]

\[
+ \frac{1}{2} \gamma^{T} D_{S} \gamma - \frac{1}{2} \left( N^{T} A_{\kappa} \varepsilon + \varepsilon^{T} A_{\kappa} N \right)
\]

\[
- \frac{1}{2} \left( M^{T} A_{\kappa} \kappa + \kappa^{T} A_{\kappa} M \right) + \frac{1}{2} \left( Q^{T} A_{Q} \gamma + \gamma^{T} A_{Q} Q \right)
\]

\[
+ \frac{1}{2} \left( N^{T} E \Lambda + \Lambda^{T} E N \right) + \frac{1}{2} \left( M^{T} B \Lambda + \Lambda^{T} B M \right)
\]

\[
+ \frac{1}{2} \left( Q^{T} G \Lambda + \Lambda^{T} G Q \right)
\]

(37)

The Hu–Washizu variational principle establishes that if the variation is taken with respect to nodal displacements and rotations \( (\Lambda) \), strains, and stresses, then all field equations of elasticity and all boundary conditions appear as Euler–Lagrange
In particular, the stationary condition for the functional, \( \delta \Pi_{\text{total}} = 0 \), enforces the following governing discretized field equation

\[
\delta N^T (E \Lambda - A_{\text{N}} \bar{e}) + \delta M^T (B \Lambda - A_{\text{M}} \bar{K}) + \delta Q^T (G \Lambda - A_{\text{Q}} \bar{\gamma}) + \delta \bar{e}^T (D_e \bar{e} + D_c \bar{K} - A_{\text{N}} N) + \delta \bar{K}^T (D_e \bar{K} + D_c \bar{e} - A_{\text{M}} M) + \delta \bar{\gamma}^T (D_s \bar{\gamma} - A_{\text{Q}} Q) + \delta \Lambda^T (E^T N + B^T M + G^T Q) - \delta \Lambda^T p = 0. \tag{38}
\]

(a) Variation of the stress resultants leads to the discrete strain-displacement relationships

\[
E \Lambda - A_{\text{N}} \bar{e} = 0 \Rightarrow \bar{e} = A_{\text{N}}^{-1} E \Lambda.
B \Lambda - A_{\text{K}} \bar{K} = 0 \Rightarrow \bar{K} = A_{\text{K}}^{-1} B \Lambda \quad \text{and} \quad \tag{39}
G \Lambda - A_{\text{Q}} \bar{\gamma} = 0 \Rightarrow \bar{\gamma} = A_{\text{Q}}^{-1} G \Lambda.
\]

(b) Variation of the strain parameters yields the discrete constitutive equations

\[
D_e \bar{e} + D_c \bar{K} - A_{\text{N}} N = 0 \Rightarrow N = A_{\text{N}}^{-1} D_e \bar{e} + A_{\text{N}}^{-1} D_c \bar{K}
D_e \bar{K} + D_c \bar{e} - A_{\text{M}} M = 0 \Rightarrow M = A_{\text{M}}^{-1} D_e \bar{K} + A_{\text{M}}^{-1} D_c \bar{e}
D_s \bar{\gamma} - A_{\text{Q}} Q = 0 \Rightarrow Q = A_{\text{Q}}^{-1} D_s \bar{\gamma}. \tag{40}
\]

(c) Variation of the nodal displacements/rotations leads to the discrete form of the equilibrium equations

\[
E^T N + B^T M + G^T Q - p = 0. \tag{41}
\]

By introducing Eqs. (39) in Eqs. (40), the parameters for the stress resultants can be expressed in terms of nodal displacements as

\[
N = A_{\text{N}}^{-1} D_e \Lambda \Lambda, \quad M = A_{\text{M}}^{-1} D_e \bar{K} + A_{\text{M}}^{-1} D_c \bar{K}, \quad Q = A_{\text{Q}}^{-1} D_s \bar{\gamma} \quad \text{and} \quad \tag{42}
\]

In a compact way, the introduction of expressions (42) into Eq. (41) yields the discrete equilibrium expressed in terms of nodal displacements and rotations as

\[
[K_{\text{extension}} + K_{\text{bending}} + K_{\text{coupling}} + K_{\text{shear}}] \Lambda = p \tag{43}
\]

Therefore, the stiffness matrix, \( K_{\Sigma_{20,20}} \), is defined by the sum of the following four terms

\[
K_{\text{extension}} = A_{\text{N}}^{-1} D_e \Lambda \Lambda, \tag{44}
K_{\text{bending}} = B^T A_{\text{M}}^{-1} D_e A_{\text{M}}^{-1} B, \tag{45}
K_{\text{coupling}} = B^T A_{\text{M}}^{-1} D_e A_{\text{M}}^{-1} E + E^T A_{\text{M}}^{-1} D_c A_{\text{M}}^{-1} B, \tag{46}
K_{\text{shear}} = G^T A_{\text{Q}}^{-1} D_s A_{\text{Q}}^{-1} G. \tag{47}
\]

3.5. The governing eigenvalue equation

The eigenvalue problem for the undamped free vibration problem takes the well-known form

\[
K u = \omega^2 M u, \tag{48}
\]

where \( K \) is the stiffness matrix of the system, \( u \) represents the eigenvectors, \( \omega \) is the natural frequency in rad/s and \( M \) is the mass matrix of the structure. The consistent element mass matrix is derived by discretizing the kinetic energy

\[
\delta U_K = \frac{1}{2} \int \rho 2 \mathbf{v} \cdot \delta \mathbf{v} \, dv, \tag{49}
\]

and by employing the displacement field defined by first-order shear deformation, the integral (49) assumes the form

\[
\delta U_K = \int \rho [\delta \bar{u}_1 \delta \bar{u}_2 \delta \bar{u}_3 \delta \bar{\phi}_1 \delta \bar{\phi}_2] \begin{bmatrix} I_1 & I_1 & I_0 & I_2 & I_2 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} dA, \tag{50}
\]

where the terms \( I_1, I_2, I_3 \) and \( \lambda \) (the contravariant components relationship) are defined by

\[
I_1 = \int_{h/2}^{h/2} \rho(z) dz, \quad I_2 = \int_{h/2}^{h/2} \rho(z) zdz, \quad I_3 = \int_{h/2}^{h/2} \rho(z) z^2 dz \tag{51}
\]

\[
\lambda = \frac{a_{12}^2}{a_{11}^2 a_{22}^2} = - \cos^2(\alpha) \tag{52}
\]

In addition, by the definition of the displacements and rotations through the shape functions, nodal displacements and nodal rotations in Eq. (25), the consistent mass matrix can be represented by

\[
\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ \text{sym} & \text{sym} & \text{sym} & \text{sym} \end{bmatrix}_{20 \times 20} \tag{53}
\]

Every \( M_i \) term of the mass matrix, where \( i \) and \( j \) represent the row and the column respectively, assumes the following form

\[
\begin{bmatrix} j_1 l_1 N_i N_j dA & j_2 a_1 N_i N_j dA & 0 & 0 \\ j_1 a_2 N_i N_j dA & j_2 a_1 N_i N_j dA & j_3 a_1 N_i N_j dA & 0 \\ 0 & j_1 a_2 N_i N_j dA & j_2 a_1 N_i N_j dA & j_3 a_1 N_i N_j dA \\ \text{sym} & \text{sym} & \text{sym} & \text{sym} \end{bmatrix}_{5 \times 5} \tag{54}
\]

Finally, we remark that all the aspects of numerical implementation associated with the above expressions are carried out by means of the commercial software package MATHEMATICA [53], which is particularly useful for the treatment of symbolic and algebraic computations.

4. Numerical results

In this section, a set of static and free vibration analyses are presented to demonstrate the applicability of the proposed finite element formulation to FG-CNTRC thin and moderately thick skew plates. Firstly, some results are compared to the limited existing bibliography, for isotropic and FG-CNTRC skew plates. Then, new bending and free vibration analyses are presented to broaden knowledge about mechanical characteristics of FG-CNTRC skew plates by taking into consideration not previously considered aspects such as asymmetric reinforcement distributions and orientations of CNTs.

4.1. Comparison studies

In order to show the validity of the proposed finite element formulation, convergence analyses are carried out in order to check the stability of the solution. Also, the free vibration results obtained by Liew et al. [54] and Zhang et al. [4] for isotropic skew plates are compared to the ones obtained by the proposed method. Then, the free vibration results for FG-CNTRC skew plates presented by Zhang et al. [4] are also verified.

4.1.1. Convergence and comparison of free vibration analysis of isotropic skew plates

In these first tests, comparison studies of free vibration analysis are carried out for isotropic skew plates with skew angles \( \alpha = 90^\circ, \ 60^\circ, \ 45^\circ \) and \( 30^\circ \), thickness-to-width ratios of \( t/b = 0.001 \) and
(thin plate) and 0.2 (moderately thick plate) and with four different kinds of boundary conditions, namely all edges simply supported (SSSS) or clamped (CCCC), and two opposite edges simply supported and the other two clamped (SCSC) or free (SFSF). The boundary conditions at any edge can be defined as follows

$$\begin{align*}
 u_x &= u_y = 0 \quad \text{Simply supported edge (S)} \\
 u_n &= u_y = u_z = 0 \quad \text{Clamped edge (C)}
\end{align*}$$

where the subscripts \( n \) and \( s \) denote the normal and tangential directions, respectively. The non-dimensional frequency parameter for vibration analysis is defined by

$$\tilde{\omega} = \frac{b^2}{\pi^2} \sqrt{\frac{\rho l}{D}}$$

where \( \omega \) is the angular frequency of the CNTRC plates, \( \rho \) is the plate density per unit volume and \( D = E t^3 / 12 (1 - \nu^2) \) is the plate flexural rigidity. A value of \( \nu = 0.3 \) for Poisson’s ratio is used for this analysis. Skew plates are characterized by the presence of stress singularities at the shell corners. Because of the simplifying assumptions commonly adopted, these problems worsen with increasing skew angle and can lead to divergent solutions. Fig. 5 shows the Von Mises stress field of fully clamped isotropic skew plates with increasing skew angles. The existence of stress concentrations at obtuse corners is highlighted. The effect of presence of these singularities on the dynamic characteristics of skew plates is well documented. For example McGee et al. [42] and Huang et al. [55] analyzed the influence of the bending stress singularities by using the Ritz method. By the implementation of comparison functions or so called corner functions the authors studied different boundary conditions and achieved great improvements in the convergence of the solution. The presence of these singularities requires the development of a convergence analysis of the dynamic characteristics in order to prove the stability of the solution. In Fig. 6, the solutions in terms of the first frequency parameter \( \tilde{\omega}_1 \) are represented for four

![Fig. 5. Corner stress singularities (Von Mises) of fully clamped (CCCC) isotropic skew plates subjected to transverse uniform loading (\( q_o = -0.1 \) MPa) and varying skew angle (\( \alpha = 90^\circ, 60^\circ, 45^\circ \) and \( 30^\circ \)).](image)

![Fig. 6. First frequency parameter \( \tilde{\omega}_1 \) convergence analysis for SSSS and SFSF isotropic skew plates in terms of mesh size \( N \times N \) (\( a/b = 1, t/b = 0.2, \alpha = 45^\circ, \psi = 0^\circ \)).](image)
for the lower modes to reach an acceptable convergence as SFSF boundary conditions. As expected, fewer elements are needed skew plate with a.

Comparison of Young’s moduli for PMMA = 300 K and Material properties of Poly (methyl methacrylate) (PMMA) at room temperature of

Table 1
Comparison study of frequency parameters (νρ²/ν²)²/ν²D of isotropic skew plates with CCCC boundary conditions (a/b = 1, t/b = 0.001, φ = 0°).

<table>
<thead>
<tr>
<th>Modes</th>
<th>Skew angle x</th>
<th>90°</th>
<th>75°</th>
<th>60°</th>
<th>45°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2
Material properties of Poly (methyl methacrylate) (PMMA) at room temperature of 300 K and (10, 10) single walled carbon nanotubes (SWCNT).

<table>
<thead>
<tr>
<th>(10, 10) SWCNT [16]</th>
<th>PMMA (T = 300 K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_{CNT} (GPa) = 5.6466 TPa</td>
<td>E’’ = 2.5 GPa</td>
</tr>
<tr>
<td>E_{11}’ (GPa) = 7.0800 TPa</td>
<td>νm = 0.34</td>
</tr>
<tr>
<td>G_{12}’ (GPa) = 1.9445 TPa</td>
<td>νm = 0.45 × 10⁻⁶/K</td>
</tr>
<tr>
<td>ν_{12}’ (GPa) = 0.175</td>
<td></td>
</tr>
</tbody>
</table>

sets of mesh sizes (8 × 8, 16 × 16, 32 × 32 and 40 × 40) and for a skew plate with a/b = 1, x = 45° and t/b = 0.2 having CCCC and SFSF boundary conditions. As expected, fewer elements are needed for the lower modes to reach an acceptable convergence as compared to the higher modes. These studies show that 24 × 24 elements are sufficient to reach accurate vibration results. Therefore, for the subsequent calculations, this mesh size is adopted.

The first eight frequency parameters for CCCC boundary conditions are presented in Table 1 together with the published results in references [54,4]. It can be seen that the present frequency parameters match very well for all cases. It is remarkable that the stiffening effect of increasing skew angles in this type of structural element is seen in all posterior results.

4.1.2. Convergence and comparison of free vibration analysis of FG-CNTRC skew plates

The next comparison analysis refers to free vibration of FG-CNTRC skew plates. A new convergence analysis is performed in order to check the suitability of the discretization for accurate

Table 3
Comparison of Young’s moduli for PMMA/CNT composites reinforced by (10, 10) SWCNT at T = 300 K with MD simulation [16].

<table>
<thead>
<tr>
<th>V_{CNT}</th>
<th>MD [16]</th>
<th>Rule of mixtures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E_{11} (GPa)</td>
<td>E_{12} (GPa)</td>
</tr>
<tr>
<td>0.12</td>
<td>94.6</td>
<td>2.9</td>
</tr>
<tr>
<td>0.17</td>
<td>138.2</td>
<td>4.9</td>
</tr>
<tr>
<td>0.28</td>
<td>224.2</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Fig. 7. First frequency parameter φ₁ convergence analysis for SSSS and SFSF FG-CNTRC skew plates in terms of mesh size N × N (UD-CNTRC, V_{CNT} = 0.12, a/b = 1, t/b = 0.001, x = 30°, φ = 0°).
predictions of this new scenario with transversely isotropic materials. The matrix Poly (methyl methacrylate), referred to as PMMA, is selected and the material properties are assumed to be \( v_m = 0.34 \), \( v_p = 45 \cdot (1 + 0.0005 \cdot \Delta T) \cdot 10^{-6} \) K and \( E_m = (3.52 – 0.0034 \cdot T) \) GPa. The armchair (10,10) SWCNTs are selected as reinforcements with properties taken from the MD simulation carried out by Shen and Zhang [16]. The material properties of these two phases are summarized in Table 2. In this study, it is assumed that the effective material properties are independent of the geometry of the CNTRC plates. The detailed material properties of PMMA/CNT for the FG-CNTRC skew plates are selected from [16] and are presented in Table 3.

In this section, the nondimensional frequency parameter \( \omega \) is defined for composites by using the matrix’s material properties as follows:

**Table 4**
Comparison study of frequency parameters \( \omega \) for a skew plate with CCCC boundary conditions \((a/b = 1, t/b = 0.001, \alpha = 0)\).

<table>
<thead>
<tr>
<th>Skew angle ( \alpha )</th>
<th>Mode</th>
<th>UD</th>
<th>FG-X</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>35.762</td>
<td>31.900</td>
<td>43.722</td>
</tr>
<tr>
<td>6</td>
<td>36.386</td>
<td>35.743</td>
<td>44.862</td>
</tr>
</tbody>
</table>

**Table 5**
Comparison study of frequency parameters \( \omega \) of uniform CNTRC skew plates of various aspect ratios with CCCC boundary conditions \((n-D)\).

<table>
<thead>
<tr>
<th>Skew angle ( \alpha )</th>
<th>Mode</th>
<th>Frequency parameters ( \omega )</th>
</tr>
</thead>
</table>

**Table 5**
Comparison study of frequency parameters \( \omega \) of uniform CNTRC skew plates of various aspect ratios with CCCC boundary conditions \((n-D)\).
\[
\dot{\omega} = \omega_0^2 \sqrt{\frac{\rho^2 t^2}{D}} \quad D = E t^3 / 12 (1 - \nu^2) \quad (57)
\]

In Fig. 7, the solutions in terms of the first frequency parameter \(\omega_0\) for the mesh sizes \((8 \times 8, 16 \times 16, 32 \times 32 \text{ and } 40 \times 40)\) are represented for a skew plate with \(a/b = 1\). \(x = 30\), UD-CNTRC, \(v_{CNTRC} = 0.12\) and \(t/b = 0.001\) having CCCC and SFSF boundary conditions. As in the previous case, a mesh pattern of \(24 \times 24\) is considered sufficient for convergence, and, henceforth this mesh size is adopted.

The results obtained by Zhang et al. [4] are compared to the ones obtained by the proposed method. Comparison of the first eight frequency parameters of the fully clamped CNTRC skew

**Table 6**

Effects of CNT volume fraction \(v_{CNTRC}\) and width-to-thickness ratio \((b/t)\) on the non-dimensional central deflection \(w_0 / t\) for CNTRC skew plates under a uniformly distributed load \(q_0 = -0.1\) MPa with SSSS boundary conditions \((a/b = 1, \varphi = 0^\circ)\).

<table>
<thead>
<tr>
<th>(V_{CNTRC})</th>
<th>(b/t)</th>
<th>(x)</th>
<th>UD</th>
<th>FG - X</th>
<th>FG - Y</th>
<th>FG - O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>ANSYS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present</td>
<td>ANSYS</td>
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<tr>
<td>Present</td>
<td>ANSYS</td>
<td></td>
<td></td>
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</tbody>
</table>

**Table 7**

Effects of CNT volume fraction \(V_{CNTRC}\) and width-to-thickness ratio \((b/t)\) on the non-dimensional central deflection \(w_0 / t\) for CNTRC skew plates under a uniformly distributed load \(q_0 = -0.1\) MPa with CCCC boundary conditions \((a/b = 1, \varphi = 0^\circ)\).

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<thead>
<tr>
<th>(V_{CNTRC})</th>
<th>(b/t)</th>
<th>(x)</th>
<th>UD</th>
<th>FG - X</th>
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Fig. 8. Variation of the non-dimensional central deflection \( w_0/t \) under uniformly distributed load \( q_0 = -0.1 \text{ MPa} \) with SSSS and CCCC boundary conditions (\( a/b = 1, V_{\text{CNT}} = 17\% \), \( b/t = 50, \phi = 0^\circ \)).

Fig. 9. Non-dimensional central axial stress \( \sigma_{xx} \) in CNTRC skew plates under a uniform load \( q_0 = -0.1 \text{ MPa} \) and various boundary conditions (\( V_{\text{CNT}} = 17\%, a/b = 1, b/t = 50, \alpha = 30^\circ, \phi = 0^\circ \)).
<table>
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<th>Mode Types</th>
<th>Skew angle $\alpha$</th>
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<th>b/t</th>
<th>Types</th>
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Table 8
Comparison study of frequency parameter $\lambda_{1}$ of skew plates with SSSS boundary conditions ($\alpha/b - 1$, $\varphi = 0^\circ$).
### Table 9
Comparison study of frequency parameter $\omega_0$ of skew plates with CCCC boundary conditions ($a/b = 1$, $\varphi = 0$).

<table>
<thead>
<tr>
<th>Skew angle $\alpha$</th>
<th>$v_{min}$</th>
<th>$b/t$</th>
<th>Mode</th>
<th>Types</th>
<th>$\omega_0$ Present ANSYS</th>
<th>$\omega_0$ Present ANSYS</th>
<th>$\omega_0$ Present ANSYS</th>
<th>$\omega_0$ Present ANSYS</th>
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$\omega_0$ is the frequency parameter, $v_{min}$ is the minimum thickness, and $b/t$ is the aspect ratio.
plates with uniform and FG-X distributions of reinforcement is given in Table 4. As previously noted, decreasing skew angles result in higher natural frequencies. Moreover, results provided by Zhang et al. [4] about the influence of the aspect ratio $t/b$ on the non-dimensional frequency parameter are also compared with the proposed approach in Table 5. Here it is also noticeable that increasing the a/b ratio decreases the frequency parameters. It can be seen that the results obtained by the proposed methodology match very well with the cited reference in both analyses. These close agreements, in combination with the convergence analyses, serve to verify the present approach and establish the foundation for its application to FG-CNTRC skew plates.

4.2. Results for FG-CNTRC skew plates

The results obtained by the proposed methodology have been shown to be stable and similar to those provided in the literature. Some new results are now presented. Here we analyze the static response of functionally graded PMMA/CNT skew plates under uniform transverse loads ($q_o$), the free vibration analysis of FG skew plates with symmetrical and unsymmetrical reinforcement distributions, and finally we show the advantages of the invariant definition of the constitutive relationships from the analysis of the influence of fiber direction on the natural frequencies. The various non-dimensional parameters used within this section are defined as

\[ \text{Non-dimensional frequency parameter: } \lambda = \frac{b^2}{t} \sqrt{\frac{\rho}{E_m}}. \]  
\[ \text{Central deflection: } \bar{w} = \frac{u_z}{t}. \]  
\[ \text{Central axial stress: } \bar{\sigma} = \frac{\sigma \cdot t^2}{|q_o| \cdot a^2}. \]

where $w_0$ is the vertical deflection at the central point. Note the non-dimensional frequency parameter is slightly different from the one employed earlier.

![Fig. 10. Effect of CNT volume fraction on the first frequency parameter $\lambda_1$ ($a/b = 1, \varphi = 0\degree$).](image-url)
4.2.1. Bending of FG-CNTRC skew plates

Several numerical examples are provided to investigate the bending analysis of FG-CNTRC skew plates under uniform transverse loading \( q_0 = -0.1 \text{ MPa} \). Four types of FG skew plates, UD-CNTRC, FG-V, FG-O and FG-X CNTRC are considered with several boundary conditions. In order to demonstrate the accuracy of the FE model used in the present study, results given by ANSYS (SHELL181, four-noded element with six degrees of freedom at each node) for the same mesh density are provided. Tables 6 and 7 shows the non-dimensional central deflection \( w/C_{22} \) for the four types of FG CNTRC skew plates subjected to a uniform transverse load \( q_0 \) with different values of width-to-thickness ratio (\( b/t = 10, 50 \)) and varying skew angles for SSSS and CCCC boundary conditions. It is noticeable that the volume fraction of the CNTs has such a large influence on the central deflection of the plates. For instance, for uniform distributions only 6% increase in the volume fraction of CNT may lead to more than 60% decrease in the central deflection. Likewise, the values of non-dimensional deflections decrease as the skew angle increases. It is also notable that the central deflections of FG-V and FG-O CNTRC plates are larger than the deflections of UD-CNTRC plates while those of the FG-X CNTRC plates are smaller. This is because the profile of the reinforcement distribution affects the stiffness of the plates. This phenomenon highlights the advantage of FG materials, in which a desired stiffness can be achieved by adjusting the distribution of CNTs along the thickness direction of the plates. It is concluded that reinforcements distributed close to the top and bottom induce higher stiffness values of plates. Fig. 8 shows the non-dimensional central deflections of skew plates with \( a/b = 1 \), \( V_{\text{CNT}} = 17\% \), \( b/t = 50 \), for SSSS and CCCC boundary conditions. The stiffening effect of higher values of skew angle \( \alpha \) can be seen clearly. Similar conclusions can be extracted from stress analysis. Fig. 9 shows the non-dimensional stress \( \sigma_{xx}/C_{22} \) distribution along the thickness for CNTRC skew plates with a skew angle of \( \alpha = 30^\circ \), subjected to a uniform transverse load \( q_0 \) with the volume fraction \( V_{\text{CNT}} = 17\% \). Due to the symmetric distribution (with respect to the mid-plane) of reinforcements for UD, FG-O and FG-X CNTRC skew plates, the central axial stress distributions is anti-symmetric. In the case of FG-V CNTRC and FG-O CNTRC skew plates, the axial stress is close to zero at the bottom and top respectively. This is because the concentration of CNTs vanishes at these points for these two distributions.

4.2.2. Free vibration analysis FG-CNTRC skew plates

The free vibration analyses of fully clamped and simply supported FG-CNTRC skew plates are given in Tables 8 and 9. The presented four types of CNTRC are considered with the CNT volume fractions of 12%, 17% and 28%. The plate geometry is defined by the following parameters, \( a/b = 1 \) and \( b/t = 10, 50 \). Here we present the first free vibration analysis of asymmetric FG-CNTRC skew plates, and the results are compared to those from the commercial

![Fig. 11. First eight mode shapes of a fully clamped UD-CNTRC skew plate for skew angles \( \alpha = 90^\circ, 60^\circ, 45^\circ \) and \( 30^\circ \). \( V_{\text{CNT}} = 12\%, a/b = 1, t/b = 0.02 \) and \( \phi = 0^\circ \).](image-url)
code ANSYS. Fig. 10 shows the evolution of the frequency parameters with varying skew angles for each reinforcement distribution separately. As expected from the previous analysis, higher skew angles give stiffer behaviors and therefore higher frequency parameters. This can be explained in terms of the plate area and the perpendicular distance between the non-skew edges. With higher skew angles, the distance between the non-skew edges decreases which increases the frequency values. Furthermore, larger volume fractions of CNTs lead to higher values of frequency parameters, due to an increase in the stiffness of the CNTRC plate when the CNT volume fraction is higher. Moreover, as could be seen in the bending simulations, we observe that the FG-X plates lead to the stiffest solutions and possess the highest frequency parameters. The explanation of this phenomenon is the same as mentioned before; reinforcements distributed closer to the extremes result in stiffer plates than those distributed nearer to the mid-plane.

Fig. 11 shows the vibration mode shapes of fully clamped UD-CNTRC plates ($V_{CNT} = 12\%$, $a/b = 1$ and $t/b = 0.02$) for skew angles $\alpha = 90^\circ$, $60^\circ$, $45^\circ$ and $30^\circ$. It is observed from these figures that mode crossing occurs as the skew angle increases.

4.2.3. Effect of direction of CNTs in FG-CNTRC skew plates on natural frequencies

Taking advantage of the invariant definition of the constitutive tensor for transversely isotropic materials, characterized by an unit vector parallel to the axis of the transverse isotropy, $\hat{n} = (\cos \varphi, \sin \varphi, 0)$, we analyze the influence of the angle $\varphi$ on the frequency parameters $\lambda$. Fig. 12 shows the variation of the first frequency parameter for several values of skew angle $\alpha$ and two boundary conditions, CCCC and SSSS. As in all the previous analyses, the frequency parameters increase for higher values of skew angle. Moreover, the results for skew angles of $\alpha = 90^\circ$ are perfectly symmetric around $\varphi = 90^\circ$. In contrast, the curves for higher skew angles present increasing levels of asymmetry. This is due to the increment of stiffness provided by fibers coupled with the stiffening effect of the skew angle. The curves can be separated into two sets divided around $\varphi = 90^\circ$. For the SSSS boundary condition set, it is clear that the increased stiffness is associated with fibers aligning the direction of the longest diagonal. Otherwise, the frequency values for the second set decreases in all cases for fiber angles above $\varphi = 90^\circ$. In the case of the CCCC boundary condition set this behavior is repeated although the maximum values are approximately obtained with fibers aligned in the horizontal Cartesian direction. This result shows the importance of taking into consideration the direction of the CNTs in order to optimize the mechanical response of the FG-CNTRC skew plates. For example, with a skew angle of $\alpha = 30^\circ$ and SSSS boundary conditions, the variation of the $\varphi$ may increase the first frequency parameter by up to 11.7% and decrease it by up to 18.6%.

5. Conclusions

In this paper, static and free vibration analyses of moderately thick FG-CNTRC skew plates are presented. An efficient finite element formulation based on the Hu–Washizu principle is presented. The shell theory is formulated in oblique coordinates and includes the effects of transverse shear strains by first-order shear deformation theory (FSDT). An invariant definition of the elastic transversely isotropic tensor based on the representation theorem is defined in oblique coordinates. Independent approximations of displacements (bilinear), strains and stresses (piecewise constant within subregions) provide a consistent mechanism to formulate four-noded skew elements with a total number of twenty degrees of freedom. A set of eigenvalue equations for the FG-CNTRC skew plate vibration is derived, from which the natural frequencies and mode shapes can be obtained.

Detailed parametric studies have been carried out to investigate...
the influences of skew angle, carbon nanotube volume fraction, plate thickness-to-width ratio, plate aspect ratio, boundary condition and distribution profile of reinforcements (uniform and three non-uniform distributions) on the static and free vibration characteristics of the FG-CNTRC skew plates. The results are compared to commercial code ANSYS and limited existing bibliography with very good agreement.

Acknowledgement

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References

[33] Zhang L, Song Z, Liew K. Nonlinear bending analysis of FG-CNT reinforced composite thick plates resting on Pasternak foundations using the element-free IMLS-Ritz method. Compos Struct 2015;128165–75.</ref>