Finite-Time Coordinated Attitude Control for Spacecraft Formation Flying Under Input Saturation

In this paper, finite-time attitude coordinated control for spacecraft formation flying (SFF) subjected to input saturation is investigated. More specifically, a bounded finite-time state feedback control law is first developed with the assumption that both attitude and angular velocity signals can be measured and transmitted between formation members. Then, a bounded finite-time output feedback controller is designed with the addition of a filter, which removes the requirement of the angular velocity measurements. In both cases, actuator saturation is explicitly taken into account, and the homogeneous system method is employed to demonstrate the finite-time stability of the closed-loop system. Numerical simulation results are presented to illustrate the efficiency of the proposed control schemes. [DOI: 10.1115/1.4029467]

1 Introduction

SFF is a promising technology for executing deep space missions, such as the distributed aperture radar, virtual co-observing, and stereo-imaging platforms for space science and Earth observation [1,2]. A formation of smaller and less-expensive spacecraft and stereo-imaging platforms for space science and Earth observations, such as the distributed aperture radar, virtual co-observing, and the virtual structure [7,8]. Ren and Beard [7] built a decentralized formation scheme based on the virtual structure approach, which is appropriate when a large number of spacecraft are involved. Liang et al. [9] developed a robust decentralized coordinated attitude control law by employing the behavioral based approach, which included both station-keeping and formation-keeping. Recently, Liu and Kumar [10] investigated formation retention problem for a spacecraft formation with communication constraints, and a robust and intelligent control law was designed to overcome communication constraints between formation members.

Despite the extensive literature on the coordinate control for SFF, a high convergence rate is still a significant requirement, and most of the existing methods ensure only asymptotic convergence. Finite-time control is an alternative way to obtain fast convergence with high accuracy and robust disturbance attenuation. Yu et al. [11] proposed a novel finite-time control strategy for robotic manipulators by employing a fast terminal sliding mode. Feng et al. [12] proposed a novel terminal sliding-mode control method to overcome the singularity problem while retaining the finite-time reachability of the systems. Du et al. [13] presented a finite-time controller for attitude tracking of a rigid spacecraft by adding a power integrator technique, and the control algorithm is further extended to solve attitude synchronization of spacecraft formation. Zhou et al. [14] used the same method to deal with attitude synchronization of spacecraft formation. The homogeneous system method is widely used to study finite-time control problems, and Bhat and Bernstein [15–17] and Hong et al. [18–20] have published extensively in this field.

In practice, precise measurements of angular velocity are not always available because of either cost limitations or implementation constraints. Thus, it is often necessary to develop attitude control algorithms in the absence of angular velocity information. Abdessameud and Tayebi [21] developed an output feedback coordinated attitude control law for spacecraft formations by introducing an auxiliary dynamical system. Zou et al. [22] proposed two robust adaptive attitude controllers based on a Chebyshev neural network in the absence of angular velocity measurements. Later, they extended their work by designing a finite-time attitude tracking controller using a terminal sliding mode and Chebyshev neural network [23]. It should be noted that output feedback control algorithms, which can stabilize the closed-loop system within a finite-time, are desirable for practical use. Recent research also includes work based on double-integrator systems [24–26], although the spacecraft formation flying problem might be less tractable. Although a finite-time observer [27,28] has been shown to provide an accurate estimate of the actual angular velocity of a single rigid spacecraft, a coordinated attitude controller for multiple spacecraft has not been presented.

Actuator saturation is another common issue that should be considered in the control system design. Due to the physical structure and energy consumption, a control saturation limit always exists, beyond which the stability of the overall system might be threatened. Also, the actuators invariably require a very high gain control output to obtain a high convergence rate. Systems using finite-time control algorithms also suffer from this drawback, especially during the initial response phase. Lu et al. [29] proposed distributed finite-time tracking control algorithms for double-integrator dynamics with bounded control inputs under a detail-balanced condition graph. The extension of these results to solve the finite-time coordinated attitude control for SFF with input saturation is nontrivial, due to the inherent nonlinearity and the coupling effect between various system states. Du and Li [30] applied homogeneous system theory to stabilize the attitude of a rigid spacecraft, but both attitude and angular velocity measurements are required in the controllers. None of the existing literature gives a solution for designing an input saturation control scheme via output feedback in the sense of finite-time stability.

Motivated by the facts and challenges stated above, in this work, a finite-time coordinated attitude control for spacecraft formation under input saturation is investigated. Initially, we propose a bounded finite-time control algorithm under the assumption that both attitude and angular velocity signals of each spacecraft are available. Based on the homogeneous system method, the closed-loop system is proven to be stabilized in finite-time, and a

Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received March 11, 2014; final manuscript received December 17, 2014; published online February 4, 2015. Assoc. Editor: Umesh Vaidya.
The attitude kinematics and dynamics of the spacecraft.

\[ \dot{q}_0 = \frac{1}{2} q_0^T \omega, \]

\[ \dot{q}_i = Q(q)_i \omega_i = \frac{1}{2} \left[ q_0 I_{3\times3} + q_i^T \right] \omega_i \]

\[ J_i \omega_i + \omega_i^T J_i \omega_i = u_i + d_i \]

where \( \dot{q}_i = [q_{i0}, \ldots, q_{in}]^T \) is the quaternion denoting the rotation from the body frame of the spacecraft to the inertial frame, and \( \omega_i \) is the angular velocity expressed in the body frame, \( J_i \) denotes the inertia tensor of the \( i \)th spacecraft, \( u_i \) is the control torque of the \( i \)th spacecraft. \( d_i \) denotes environment disturbance acting on \( i \)th spacecraft. For a given vector \( \theta = (\theta_1, \theta_2, \theta_3)^T \), \( \theta^\circ \) is defined as

\[ \theta^\circ = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}. \]

Equations (1a)–(1c) may be combined to give the nonlinear dynamic equation

\[ J_i^T(q_i) \ddot{q}_i + C_i^T(q_i, \dot{q}_i) \dot{q}_i = u_i^T + d_i^T, \]

where \( J_i^T(q_i) = F_i^T J_i F_i, \ C_i^T(q_i, \dot{q}_i) = F_i^T (F_i \dot{q}_i)^T J_i - J_i F_i \dot{Q}_i F_i, \) \( u_i^T = F_i^T u_i, \) \( d_i^T = F_i^T d_i, \) and \( F_i(q_i) = Q_i(q_i)^{-1}. \) Hence, one has the following properties:

**Property 1.** The matrix \( J_i^T(q_i) - 2C_i^T(q_i, \dot{q}_i) \) is skew-symmetric for all \( q_i \) [32]. Hence

\[ x^T \left( J_i^T(q_i) - 2C_i^T(q_i, \dot{q}_i) \right) x = 0, \quad \forall x, q_i \in \mathbb{R}^n \]

**Remark 1.** In order to ensure the existence of \( F_i(q_i), \) the matrix \( Q_i(q_i) \) should be invertible. This implies that the initial state must be guaranteed to the matrix \( \dot{Q}(q(0)) = (1/2)q_0(0) \neq 0, \) and the control scheme is designed to guarantee that \( \dot{q}_0(t) \neq 0 \) for all time.

**Assumption 1.** To simplify the theoretical analysis, we neglect the external disturbance, i.e., \( d(t) = 0. \)

2.2 Graph Theory. We introduce graph theory [33] to describe the communication links between spacecraft in the formation. Let \( G = (N,E,A) \) be a graph, in which \( N = \{n_1, n_2, \ldots, n_n\} \) is a finite nonempty set of nodes and \( E \subseteq N \times N \) is a set of ordered pairs of nodes, called edges. An edge \((n_i, n_j) \in E \) denotes that node \( n_i \) can obtain information from node \( n_j; \) thus, \( n_i \) is a parent of \( n_j, \) and \( n_j \) is a child of \( n_i. \) Then, the set of the neighbors of node \( n_i \) is denoted by \( N_i = \{j : (n_i, n_j) \in E\}. \) If a graph has the property that \((n_i, n_j) \in E \Leftrightarrow (n_j, n_i) \in E, \) the graph is called an undirected graph. An adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) associated with \( G \) represents the communication between each node, in which \( a_{ij} > 0 \) if \((n_i, n_j) \in E, \) while \( a_{ij} = 0 \) otherwise. The Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{n \times n} \) associated with \( G \) is defined as \( l_{ij} = \sum_{k=0}^{n} a_{ik} \) if \( i = j, \) otherwise \( a_{ij} = 0. \) A graph \( G \) is said to be connected if there exists a path between each pair of distinct nodes. Throughout this paper, the spacecraft formation with \( n \) spacecraft is considered, where each spacecraft is represented by a node in the graph. If a leader exists, it is denoted by node \( 0, \) and the other spacecraft are called the followers of the leader. Let the diagonal matrix \( B = \text{diag} [a_{01}, \ldots, a_{0n}] \) be the connection weight between the leader and the followers, where \( a_{0i} > 0 \) if the \( i \)th follower has access to the leader, and otherwise \( a_{0i} = 0. \)

**Lemma 1.** \( M \) is a symmetric and positive definite matrix, if Assumption 2 holds [34].

2.3 Definitions and Lemmas. The following definitions and lemma are further required for system analysis.

Consider the following system:

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is non-Lipschitz continuous on an neighborhood \( D \) of the origin \( x = 0 \) in \( \mathbb{R}^n. \)

**Definition 1.** The equilibrium \( x = 0 \) of (3) is finite-time convergent if there is an open neighborhood \( U \) of the origin and a function \( T_x : U \setminus \{0\} \to (0, \infty), \) such that every solution trajectory \( x(t, x_0) \) of (3) starting from the initial point \( x_0 \in U \setminus \{0\} \) is well-defined and unique in forward time for \( t \in [0, T_x(x_0)) \), and \( \lim_{t \to T_x(x_0)} x(t, x_0) = 0. \) Consider the settling time \( T_x(x_0) \) is called the settling time of the initial state \( x_0. \) The equilibrium of (3) is finite-time stable if it is Lyapunov stable and finite-time convergent. If \( U = D = \mathbb{R}^n \), the origin is a globally finite-time stable equilibrium [18].

**Definition 2.** Let \( (r_1, \ldots, r_I)^T \in \mathbb{R}^I \) with \( r_i > 0, i = 1, \ldots, n. \) Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. \( V \) is said to be homogeneous of degree \( \alpha \) with respect to \( (r_1, \ldots, r_I) \) if for any given \( \varepsilon > 0 \)

\[ V(\varepsilon^r x_1, \ldots, \varepsilon^r x_n) = \varepsilon^V(x), \quad \forall x \in \mathbb{R}^n \]

Let \( f : \mathbb{R}^n \to \mathbb{R}^n, \) \( f = (f_1(x), \ldots, f_n(x))^T \) be a continuous vector field. \( f(x) \) is said to be homogeneous of degree \( \kappa \in \mathbb{R} \) with respect to \( (r_1, \ldots, r_I) \) for any given \( \varepsilon > 0, \)

\[ f(\varepsilon^r x_1, \ldots, \varepsilon^r x_n) = \varepsilon^\kappa f(x), \quad i = 1, \ldots, n, \forall x \in \mathbb{R}^n \]

System (3) is said to be homogeneous if \( f(x) \) is homogeneous.

**Lemma 2.** Consider the following system

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \]

Suppose that the origin of system (4) is asymptotically stable and the system is homogeneous of degree \( \kappa. \) If \( \kappa < 0, \) then the origin of system (4) is finite-time stable [18].

**Lemma 3.** Consider the system

\[ \dot{x} = f(x) + \tilde{f}(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \]
where \( f(x) \) is a continuous homogeneous vector field of \( \kappa < 0 \) with respect to \((r_1, r_2, \ldots, r_p)\), and \( f \) satisfies \( f(0) = 0 \). Assume \( x = 0 \) is an asymptotically stable equilibrium of the system \( \dot{x} = f(x) \). Then \( x = 0 \) is a locally finite-time stable equilibrium of system (5) if

\[
\lim_{t \to 0} \frac{f(x_1, x_2, \ldots, x_p)}{x_1^n + x_2^n + \cdots + x_p^n} = 0, \quad g = 1, \ldots, p, \quad \forall x \neq 0.
\]  

(6)

Moreover, if the stable equilibrium \( x = 0 \) of the original system (5) is globally asymptotically stable, then \( x = 0 \) is a globally finite-time stable equilibrium of system (5).

**Lemma 4.** Consider the autonomous system

\[
\dot{x} = f(x)
\]  

(7)

where \( f : \mathbb{R}^p \to \mathbb{R}^p \) is a continuous function with \( D \subset \mathbb{R}^p \). Let \( \Omega \subset D \) be a compact set that is positively invariant with respect to (7). Let \( V : D \to \mathbb{R} \) be a continuously differentiable function such that \( V(x) \leq 0 \) in \( \Omega \). Let \( E \) be the set of all points in \( \Omega \) where \( V(x) = 0 \). Let \( \Omega \) be the largest invariant set in \( E \). Then every solution starting in \( \Omega \) approaches \( \Omega \) as \( t \to \infty \) [34].

3 Main Results

The finite-time coordinated attitude control problem for spacecraft formation flying is studied. Two distributed bounded control schemes with and without angular velocity information are developed, under which each spacecraft can be steered to the reference attitude \( q_0 \), which acts as a virtual leader, while the angular deviation between them can be stabilized to zero. Based on the homogeneous system theory, a finite-time full-state feedback control law with input saturation is first designed. Then, combined with a filter, a bounded finite-time control law via output feedback is proposed.

3.1 Bounded Finite-Time Coordinated Attitude Control via Full-State Feedback. In this section, we propose a finite-time coordinated attitude control law with input saturation under the assumption that both attitude and angular velocity signals are available. By employing a hyperbolic tangent function, the actual control output signal is naturally bounded.

**Theorem 1.** Consider the spacecraft formation system governed by Eq. (2) with a stationary virtual leader, where Assumptions 1 and 2 hold and at least one spacecraft has access to the virtual leader. The closed-loop system can be stabilized in finite-time under the following control law:

\[
u_i = Q_i^T(q_i)\left\{-h_{pi} \sum_{j=0}^{n} a_{ij} \tanh(\text{sign}(q_i - q_j)) - h_{di} \tanh(\text{sign}(q_i))\right\}
\]  

(8)

where \( 0 < z_1 < 1, \quad x_2 = (2z_1)/(x_1 + 1) \), and with control gains \( h_{pi} > 0, \quad h_{di} > 0 \). The upper bound of control torque can be calculated as

\[
\|\nu\| \leq \frac{\|Q_i(\|q_i\|)\|_2}{h_{pi}} \sum_{j=0}^{n} a_{ij} \tanh(\text{sign}(q_i - q_j))
\]

\[
+ \|Q_i(\|q_i\|)\|_{h_{di}} \tanh(\text{sign}(q_i))\|
\]

\[
= \sqrt{3} h_{pi} \sum_{j=0}^{n} a_{ij} + h_{di} \leq u_{\max}
\]  

(9)

**Proof.** Redefine the system state variables as \( x_{ii} = q_i \) - \( q_0 \), \( x_2 = q_i \), where \( q_0 \) is considered to be a constant. Thus, Eq. (2) is transformed into the form

\[
\dot{x}_i = x_2 
\]  

(10a)

\[
J_i^{T}(x_{ii} + q_0) x_2 + C_i^{T}(x_{ii} + q_0, x_2) x_2 = u_i^* 
\]  

(10b)

 Submitting the control law (8) into system (10) yields the closed-loop system (11)

\[
\dot{x}_i = x_2 
\]  

(11a)

\[
J_i^{T}(x_{ii} + q_0) x_2 + C_i^{T}(x_{ii} + q_0, x_2) x_2 = -h_{pi} \sum_{j=0}^{n} a_{ij} \tanh(\text{sign}(x_{ii} - x_j))
\]

\[
- h_{di} \tanh(\text{sign}(x_{ii}))
\]  

(11b)

Noticing that system (11) is not homogeneous, the following analysis is divided into two steps to prove system (11) can be stabilized in finite-time:

**Step 1:** Global asymptotic stability

Consider a candidate Lyapunov function

\[
V_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} \frac{1}{2} h_{ji} a_{ij} \tanh(\text{sign}(x_{ij} - x_{jm})) (x_{2i} - x_{2m})
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{2} h_{ji} a_{ij} \tanh(\text{sign}(x_{ii})) x_{2i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{2} h_{ji} a_{ij} \tanh(\text{sign}(x_{ii})) x_{2i} + \frac{1}{2} x_i^T J_i^{T}(x_{ii} + q_0) x_2
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ji} a_{ij} \tanh(\text{sign}(x_{ij} - x_{jm})) (x_{2i} - x_{2m})
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ji} a_{ij} \tanh(\text{sign}(x_{ii})) x_{2i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ji} a_{ij} \tanh(\text{sign}(x_{ii})) x_{2i}
\]

\[
- \frac{1}{2} h_{ji} a_{ij} \tanh(\text{sign}(x_{ii})) x_{2i}
\]

\[
+ \frac{1}{2} x_i^T J_i^{T}(x_{ii} + q_0) x_2 + \frac{1}{2} \sum_{i=1}^{n} x_i^T J_i^{T}(x_{ii} + q_0) x_2
\]

(12)

Obviously, \( V_1 \) is positive definite, as shown in Ref. [19], and differentiating \( V_1 \) along system (12) gives

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Noting the fact that the communication topology of the spacecraft formation is a directed graph and that 
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) \] 
forms a directed graph, we have:
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right)
\]

Then, by employing Property 1, one has
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) = 0.
\]

Thus, we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) = 0.
\]

Let \( \Omega = \{ (x_{1i}, x_{2i}) : V_1 = 0 \} \) and \( \bar{\Omega} \) be the largest invariant set in \( \Omega \). Thus, we know that \( V_1 \equiv 0 \) on \( \Omega \) implies \( x_{2i} = 0 \) and \( \bar{x}_{2i} = 0 \). By recalling Eq. (11b), one obtains
\[
- \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) - h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i}) \right) = 0
\]

Hence, \( \lim_{t \to 0} x_{1i} = 0 \) and \( \lim_{t \to 0} x_{1i} - x_{1j} = 0 \), which implies that \( \lim_{t \to 0} q_i - q_j = 0 \) and \( \lim_{t \to 0} q_i - q_j = 0 \). Then, the closed-loop system (11) achieves asymptotically stable.

Step 2: Local finite-time stability

Since the closed-loop system (11) is not homogeneous, in order to appeal to Lemma 3, Eq. (11) should be rearranged. Recall that \( \tanh \left( \text{sign} (x) \right) = \text{sign} (x)^3 + o \left( \text{sign} (x)^3 \right) \) around the origin [35]. Then, Eq. (11) is transformed as
\[
\dot{x}_{1i} = x_{2i}
\]

(17a)

\[
\dot{x}_{2i} = J_i(q_0)^{-1} \left\{ -h_{ij} \sum_{j=1}^{n} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) + h_{ij} a_{ij} \right\} + \bar{g}_2(x_{1i}, x_{2i})
\]

(17b)

where
\[
\bar{g}_2(x_{1i}, x_{2i}) = -J_i(x_{1i} + q_0)^{-1} C_i(x_{1i} + q_0, x_{2i}) - \left( J_i(x_{1i} + q_0)^{-1} - J_i(q_0)^{-1} \right) \left[ h_{ij} \sum_{j=1}^{n} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) + h_{ij} a_{ij} \tanh (x_{1i} x_{1j}) \right]
\]

(18)

It is easy to verify that \( \bar{g}_2(0) = 0 \). A reduced system of (11) can be written as
\[
\dot{x}_{1i} = x_{2i}
\]

(19a)

\[
\dot{x}_{2i} = J_i(q_0)^{-1} \left\{ -h_{ij} \sum_{j=1}^{n} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right) + h_{ij} a_{ij} \right\}
\]

(19b)

System (19) is homogeneous of degree \( k = (z_1 - 1)/(z_1 + 1) \) with respect to \( (r_2, r_1, r_1) \), where \( r_1 = (2)/(z_1 + 1) \), \( r_2 = 1 \), \( 0 < z_1 < 1 \), \( x_2 = (2x_1)/(z_1 + 1) \) is in \( (0.5, 1) \). Next, we show that \( \lim_{t \to 0} \left( \bar{g}_2(x_{1i}, x_{2i}) \right) \) is equal to the mean value inequality, and following a similar analysis to that in Ref. [36], one has
\[
\lim_{t \to 0} \frac{\bar{g}_2(x_{1i}, x_{2i})}{x_{1i}^r x_{2i}^r} = - \lim_{t \to 0} \frac{J_i(q_0)^{-1} C_i(x_{1i} + q_0, x_{2i})}{x_{1i}^r x_{2i}^r} \left( J_i(q_0)^{-1} - J_i(q_0)^{-1} \right) \left[ h_{ij} \sum_{j=1}^{n} a_{ij} \tanh (x_{1i} x_{1j}) \right]
\]

(20)

We now construct the following candidate Lyapunov function:
\[
V_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{n} h_{ij} a_{ij} \tanh \left( \text{sign} (x_{1i} - x_{1j}) \right)
\]

(21)
system (11) is finite-time stable. Since the equilibrium control law with a filter is derived. Thus, the attitude of the spacecraft will be shown by numerical simulation. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure. However, the disturbance will be out of the scope of the theoretical analysis procedure.

Remark 2. Until now, among research on finite-time control using homogeneous system theory, only a qualitative analysis of the fast finite-time convergence rate has been presented, and none of these works calculated the convergence time quantitatively. The explicit formulation of the convergence time remains an interesting problem.

Remark 3. It can be seen from the structure of the control law (8) that the exact inertia matrix is not necessarily needed. Hence, controller (8) has good robustness to uncertainties. Also, the existing literature that discusses finite-time stable systems by employing homogeneous system theory has not solved the external disturbance problem, due to the limitation of this theory. Inevitably, the external disturbance has not been taken into consideration directly throughout the above theoretical analysis procedure. However, the disturbance rejection ability will be shown by numerical simulation.

3.2 Bounded Finite-Time Coordinated Attitude Control via Output Feedback. Often the angular velocity of spacecraft is difficult to obtain in some engineering circumstances. To handle the lack of angular velocity measurement, an output feedback control law with a filter is derived. Thus, the attitude of the spacecraft in the formation can align to the virtual leader in a finite-time.

**Theorem 2.** Consider the spacecraft formation system governed by Eq. (2) with a stationary virtual leader, where Assumptions 1 and 2 hold and at least one spacecraft has access to the virtual leader. The following bounded control law (23) combined with filter (24), where $0 < \alpha_i < 1$, $\alpha_2 = (\alpha_1 + 1)/2$, and with control gains $k_i > 0$, $k_2 > 0$, $k_{pi} > 0$, and $k_{pi} > 0$, can guarantee the closed-loop stability in finite-time

$$u_i = Q_i^r(q_i) \left\{ -k_{pi} \sum_{j=0}^{n} a_j \tanh[\text{sig}(q_i - q_j)] - k_{pi} \tanh[\text{sig}(y_i)] \right\} \tag{23}$$

$$\dot{y}_i = -k_i \tanh[\text{sig}(y_i)] + k_2 \dot{q}_i \tag{24}$$

The upper bound of the control torque can be calculated as

$$\|u_i\| \leq \|Q_i^r(q_i)\| \kappa_i \sum_{j=0}^{n} a_j \tanh[\text{sig}(q_i - q_j)]^2 + \|Q_i^r(q_i)\| \kappa_i \tanh[\text{sig}(y_i)]^2$$

$$= \sqrt{3} \kappa_i \sum_{j=0}^{n} a_j + k_2 \leq u_{\text{max}} \tag{25}$$

**Proof.** Redefine the system state variables as $x_{i1} = q_i - q_0$, $x_{i2} = \dot{q}_i$, $x_{i3} = y_i$. Then the closed-loop system can be written as

$$\dot{x}_{i1} = x_{i2} \tag{26a}$$

$$J_i(x_{i1} + q_0)\dot{x}_{i2} + C_i(x_{i1} + q_0, x_{i2})x_{i3} = u_i \tag{26b}$$

$$\dot{x}_{i3} = -k_i \tanh[\text{sig}(x_{i3})] + k_2 x_{i2} \tag{26c}$$

Submitting the control law (23) and filter (24) into system (26) yields the closed-loop system

$$\dot{x}_{i1} = x_{i2} \tag{27a}$$

$$J_i(x_{i1} + q_0)\dot{x}_{i2} + C_i(x_{i1} + q_0, x_{i2})x_{i3} = -k_{pi} \sum_{j=0}^{n} a_j \tanh[\text{sig}(x_{i1} - x_{j1})]^2$$

$$+ k_{pi} \tanh[\text{sig}(x_{i3})]^2 - k_{pi} \tanh[\text{sig}(y_i)]^2$$

$$\dot{x}_{i3} = -k_i \tanh[\text{sig}(x_{i3})] + k_2 x_{i2} \tag{27c}$$

In the following, we will prove that system (27) is finite-time stable. It can be seen that system (27) is not homogeneous, and thus two steps are taken to accomplish the proof.

Step 1: Global asymptotic stability

Consider a candidate Lyapunov function

$$V_3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} \frac{1}{k_{pj}} \int_{0}^{x_{i1}(m) - x_{j1}(m)} a_j \tanh[\text{sig}(x_{i1}(m) - x_{j1}(m))]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} a_j \tanh[\text{sig}(x_{i1}(m))]$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{pj} \tanh[\text{sig}(x_{i1}(m))]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} \frac{1}{k_{pj}} \int_{0}^{x_{i3}(m)} \tanh[\text{sig}(x_{i3}(m))]$$

Obviously, $V_3$ is positive definite, and differentiating $V_3$ along the system (27) gives
Table 1  Initial system state

| Spacecraft 1 | $\mathbf{\ddot{q}_1}(0) = [0.5477 \ 0.3 \ -0.5 \ 0.6]^T$ | $\mathbf{\omega}_1(0) = [0.1 \ 0.15 \ -0.2]^T$ |
| Spacecraft 2 | $\mathbf{\ddot{q}_2}(0) = [0.8888 \ -0.2 \ 0.1 \ 0.4]^T$ | $\mathbf{\omega}_2(0) = [0.15 \ -0.2 \ -0.1]^T$ |
| Spacecraft 3 | $\mathbf{\ddot{q}_3}(0) = [0.6856 \ 0.4 \ -0.6 \ 0.1]^T$ | $\mathbf{\omega}_3(0) = [0.2 \ -0.3 \ 0.1]^T$ |
| Spacecraft 4 | $\mathbf{\ddot{q}_4}(0) = [0.9274 \ 0.2 \ -0.3 \ 0.1]^T$ | $\mathbf{\omega}_4(0) = [-0.1 \ 0.2 \ -0.2]^T$ |

Fig. 2  The absolute attitude error under controller (8) in the absence of disturbance ($\varepsilon_1 = 0.5$)

Fig. 3  The angular velocity under controller (8) in the absence of disturbance ($\varepsilon_1 = 0.5$)
\[ \dot{V}_3 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{ij} a_{ij} \tanh \left( \text{sign} \left( x_{1(m)} - x_{1(j)} \right) \right) \left( x_{2(m)} - x_{2(j)} \right) \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{ij} a_{ij} \tanh \left( \text{sign} \left( x_{1(m)} \right) \right) \left( x_{2(m)} \right) + \frac{1}{2} \sum_{i=1}^{n} \sum_{m=1}^{3} k_{ij} \tanh \left( \text{sign} \left( x_{3(m)} \right) \right) x_{3(m)} \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{ij} a_{ij} \tanh \left( \text{sign} \left( x_{1(m)} - x_{1(j)} \right) \right) \]
\[ - \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{ij} a_{ij} \tanh \left( \text{sign} \left( x_{1(m)} \right) \right) \]
\[ - \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{ij} a_{ij} \tanh \left( \text{sign} \left( x_{3(m)} \right) \right) \]
\[ \times \left[ -k_1 \tanh \left( \text{sign} \left( x_{3i} \right) \right) + k_2 x_{2j} \right] \leq 0 \]  
\[ (29) \]

Fig. 4 The control torque under controller (8) in the absence of disturbance \((x_1 = 0.5)\)

Fig. 5 The absolute attitude error under controller (8) in the absence of disturbance \((x_1 = 1)\)
Let \( \Omega = \{(x_{1i}, x_{2i}, x_{3i}) : V_3 = 0\} \) and \( \check{\Omega} \) be the largest invariant set in \( \Omega \). Thus, we know that \( \dot{V}_3 \equiv 0 \) on \( \Omega \), which implies \( x_{3i} = 0 \) and \( \dot{x}_{3i} = 0 \). Thus, from the equation \( \dot{y}_i = -k_1 \tanh[\text{sign}(x_{3i})^{\frac{x_3}{k_2}}] + k_2 x_{2i} \), it can be easily concluded that \( x_{3i} \equiv 0 \). From Eq. (27b), we get \( -k_{pl} \sum_{j=1}^{n} a_{ij} \tanh[\text{sign}(x_{3i} - x_{1j})^{\frac{x_1}{k_2}}] + k_{pl} \tanh[\text{sign}(x_{3i})^{\frac{x_3}{k_2}}] \equiv 0 \). Hence, \( \lim_{i \to 0} x_{1i} - x_{1j} = 0 \), \( \lim_{i \to 0} x_{1i} = 0 \). Hence, system (27) achieves asymptotic stability.

Step 2: Local finite-time stability
Because system (27) is not homogeneous, we have to transform it into the system

\[
\begin{align*}
\dot{x}_{1i} &= x_{2i} \\
\dot{x}_{2i} &= - (J_i'(q_0))^{-1} \left[ k_{pi} \sum_{j=1}^{n} a_{ij} \text{sign}(x_{1i} - x_{1j})^{\frac{x_1}{k_2}} \right. \\
&\quad \left. + k_{pi} a_{ij} \text{sign}(x_{1i})^{\frac{x_1}{k_2}} + k_{pi} a_{ij} \text{sign}(x_{1i})^{\frac{x_3}{k_2}} + f_{2i}(x_{1i}, x_{2i}, x_{3i}) \right]
\end{align*}
\]

where

\[
\begin{align*}
\dot{x}_{3i} &= -k_1 \text{sign}(x_{3i})^{\frac{x_3}{k_2}} + k_2 x_{2i} + f_{3i}(x_{1i}, x_{2i}, x_{3i})
\end{align*}
\]

Fig. 6 The angular velocity under controller (8) in the absence of disturbance \((x_1 = 1)\)

Fig. 7 The control torque under controller (8) in the absence of disturbance \((x_1 = 1)\)
\[ f_3(x_1, x_2, x_3) = -k_1 \sigma(x_3)^{a_1} \]  

(31b)

Obviously, \( f_3(0) = 0 \) and \( f_3(0) = 0 \). A reduced system of (30) can be written as

\[ x_{1i} = x_{2i} \]  

(32a)

\[ x_{2i} = - (J_i^T(q_i))^{-1} \left[ k_{p1} \sum_{j=1}^{n} a_{ij} \sigma(x_{ij} - x_{ij})^{a_{ij}} + k_{p2} \sigma(x_{2i})^{a_{2i}} + k_{p3} \sigma(x_{3i})^{a_{3i}} \right] \]  

(32b)

\[ x_{3i} = -k_1 \sigma(x_{3i})^{a_1} + k_2 x_{2i} \]  

(32c)

It can be easily verified that system (32) is homogeneous of degree \( k = a_1 - 1 \) with respect to \( (r_1 1_{n}, r_2 1_{n}, r_3 1_{n}) \), where \( r_1 = r_3 = 2, r_2 = a_1 + 1, and 0 < a_1 < 1 \). Then

**Fig. 8** The absolute attitude error under controller (23)/(24) in the absence of disturbance (\( x_1 = 0.6 \))

**Fig. 9** The control torque under controller (23)/(24) in the absence of disturbance (\( x_1 = 0.6 \))

**Table 2** Performance comparison under controller (8) with different \( x_1 \)

<table>
<thead>
<tr>
<th>x_1</th>
<th>Convergence time (s)</th>
<th>Attitude error (( q_{ei} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>23</td>
<td>2 x 10^{-1}</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>2 x 10^{-2}</td>
</tr>
</tbody>
</table>

Improvement (%)

<table>
<thead>
<tr>
<th></th>
<th>23</th>
<th>30</th>
<th>Improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attitude error</td>
<td>90</td>
<td>90</td>
<td>0</td>
</tr>
</tbody>
</table>
\[
\lim_{\epsilon \to 0} \frac{\hat{f}_2(e^\epsilon x_{i1}, e^\epsilon x_{i2}, e^\epsilon x_{i3})}{e^{k+r_2}} = -\lim_{\epsilon \to 0} \frac{(J'_1(e^\epsilon x_{i1} + q_0))^{-1} C'_i(e^\epsilon x_{i1} + q_0, e^\epsilon x_{i2}) e^\epsilon x_{i2}}{e^{k+r_2}} - \lim_{\epsilon \to 0} \frac{(J'_1(e^\epsilon x_{i1} + q_0))^{-1} - (J'_1(q_0))^{-1}}{e^{k+r_2}}
\times \left\{ k_{pi} \sum_{j=1}^{n} a_i \tanh[\text{sig}(e^\epsilon x_{i1} - e^\epsilon x_{i1})^{z_i}] + k_{pi} a_i \tanh[\text{sig}(e^\epsilon x_{i1})^{z_i}] + k_{i0} \tanh[\text{sig}(e^\epsilon x_{i1})^{z_i}] \right\}  \\
+ \lim_{\epsilon \to 0} \frac{(J'_1(q_0))^{-1}}{e^{k+r_2}} \left\{ k_{pi} \sum_{j=1}^{n} a_i \text{sig}(x_{i1} - x_{i1})^{z_i} + k_{pi} a_i \text{sig}(x_{i1})^{z_i} + k_{j0} \text{sig}(x_{i1})^{z_i} \right\} = 0
\]  

(33)

\[
\lim_{\epsilon \to 0} \frac{\hat{f}_3(e^\epsilon x_{i1}, e^\epsilon x_{i2}, e^\epsilon x_{i3})}{e^{k+r_3}} = -\lim_{\epsilon \to 0} \frac{k_i \text{sig}(e^\epsilon x_{i3})^{z_i}}{e^{k+r_3}} = 0
\]  

(34)

Fig. 10 The absolute attitude error under controller (23)/(24) in the absence of disturbance \((x_1 = 1)\)

Fig. 11 The control torque under controller (23)/(24) in the absence of disturbance \((x_1 = 1)\)
Construct the following candidate Lyapunov function:

\[ V_4 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{pj} \int_{0}^{x_{(j(m)}} a_i \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

\[ + \sum_{i=1}^{n} \sum_{m=1}^{3} k_{pj} \int_{0}^{x_{(j(m)}} a_i \text{sign}(x_{(i(m)} - x_{(j(m)})^2 + \frac{1}{2} \sum_{i=1}^{n} x_{(i(m)}^T(J_i(q) + x_{(j(m)})^2 \]

\[ + \sum_{i=1}^{n} \sum_{m=1}^{3} \frac{k_{pj}}{2} \int_{0}^{x_{(j(m)}} \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

(35)

Taking the derivative of \( V_4 \), along the reduced system (32), yields

\[ \dot{V}_4 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{3} k_{pj} \dot{a}_i \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

\[ + \sum_{i=1}^{n} \sum_{m=1}^{3} k_{pj} \dot{a}_i \text{sign}(x_{(i(m)} - x_{(j(m)})^2 + \sum_{i=1}^{n} \sum_{m=1}^{3} \frac{k_{pj}}{2} \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

\[ - \sum_{i=1}^{n} \sum_{m=1}^{3} \frac{k_{pj}}{2} \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

\[ - \sum_{i=1}^{n} \sum_{m=1}^{3} \frac{k_{pj}}{2} \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

\[ = - \sum_{i=1}^{n} \sum_{m=1}^{3} \frac{k_{pj}}{2} \text{sign}(x_{(i(m)} - x_{(j(m)})^2 \]

(36)

A similar procedure also proves the asymptotic stability of system (32). Moreover, with the analysis in Step 2 and Lemma 2, system (32) can be stabilized in a finite-time at the equilibrium \( x_{(i)} = x_{(j)} = x_{(m)} = 0 \).

According to Lemma 3, \( x_{(i)} = x_{(j)} = x_{(m)} = 0 \) is also the finite-time stable equilibrium of system (27), which implies that \( q_i \to q_{i0}, \, q_j \to 0, \, q_m \to 0 \) in a finite-time. Thus, the proof of Theorem 2 is completed.

Remark 4. It should be mentioned that, although \( \dot{q}_i \) is required to implement the filter (24), angular velocity is not necessary to calculate control torque. This is because the output of the filter, i.e., \( y_i \), comes from the integration of Eq. (24). Thus, in the output control law (23), the information of \( q_i \) is used rather than \( \dot{q}_i \).

Remark 5. Obviously, in both control laws, if we select \( x_1 = x_2 = 1 \), controller (8) and controller (23) reduce to asymptotic control schemes. The detailed theoretical demonstration is given in Step 1 of Theorems 1 and 2, which we do not restate here. Simulation examples will be presented in Sec. 4.

### Table 3 Performance comparison under controller (23)/(24) with different \( z_1 \)

| \( z_1 \) | Convergence time (s) | Attitude error \( |q_{i0}| \) | Improvement (%) |
|----------|----------------------|--------------------------|-----------------|
| 0.6      | 26                   | \( 6 \times 10^{-3} \)   | 35              |
| 1        | 40                   | \( 1 \times 10^{-2} \)   | 40              |

Fig. 12 The absolute attitude error under controller (8) in the presence of disturbance \( (x_1 = 0.5) \)
The initial attitude and initial angular velocity of each spacecraft are given in Table 1. Their inertia matrices are chosen as:
\[ J_1 = \text{diag} \{ 11.24, 12.58, 10.34 \} \text{kg} \cdot \text{m}^2, \quad J_2 = \text{diag} \{ 10.77, 11.58, 11.32 \} \text{kg} \cdot \text{m}^2, \quad J_3 = \text{diag} \{ 10.35, 10.97, 10.52 \} \text{kg} \cdot \text{m}^2, \quad \text{and} \quad J_4 = \text{diag} \{ 10.35, 12.22, 11.16 \} \text{kg} \cdot \text{m}^2. \]

And the actual control torque is limited not to exceed \( u_{\max} = 2 \text{N} \cdot \text{m} \).

In the following, we first show the simulation results of the two control schemes in the absence of external disturbance. Then, to examine the disturbance rejection performance of the proposed controllers, we assume the disturbance torque to be:
\[ d_i(t) = \begin{bmatrix} 0.12 \sin(0.1t) & 0.06 \cos(0.15t) & 0.08 \sin(0.2t + 1) \end{bmatrix}^T \text{N} \cdot \text{m}. \]

First, the bounded full-state feedback finite-time coordinated attitude controller (8) is adopted for the spacecraft formation flying, and control parameters are selected to be \( \alpha_1 = 0.5, \alpha_2 = 2/3, k_{\mu} = 0.8, \text{ and } k_{di} = 1.4, i = 1, 2, 3, 4. \) Figure 2 shows the absolute attitude errors between different spacecraft and the reference attitude, and the corresponding angular velocity is shown in Fig. 3. It is observed that the attitude errors are reduced to approximately zero with zero final angular velocity within 23 s. The performance of the closed-loop system is highly accurate, such that the relative attitude error is \( q_{ei} < 2 \times 10^{-3} \). The actual control torque under control law (8) is shown in Fig. 4, which highlights the torque is limited to \( |u_{\max}| = 2 \text{N} \cdot \text{m} \). Figures 5–7 present the performance of the system under controller (8) with the choice \( x_1 = 1 \), which guarantees the system is asymptotically stable. The settling time is longer while the accuracy is lower. From Figs. 4 and 7, we see the magnitude of the control torque in each case is quite similar, which ensures the comparison is fair. The detailed comparison of the performance with different values of \( x_1 \) is given in Table 2.

Figures 8 and 9 show the response of the system under the finite-time controller (23) with filter (24), and the control...
parameters are selected to be $z_1 = 0.6$, $k_{ij} = 1$, $k_{ii} = 3$, $k_{ii} = 0.2$, $k_{ii} = 0.4$, and $i = 1, 2, 3, 4$. From Fig. 8, it can be observed that the attitude of each spacecraft aligned to the reference attitude in the absence of angular velocity measurements in less than 26 s with a very high accuracy. The corresponding control torque is shown in Fig. 9, which is also bounded by $|u_{\text{max}}| = 2 \text{ N} \cdot \text{m}$. When we choose $z_1 = 1$, the control law (23)/(24) can ensure the system achieves asymptotic stability. The attitude error and control torques are shown in Figs. 10 and 11. Table 3 presents the detailed comparison of the performance with different $z_1$.

The following simulations will illustrate the robustness of the proposed controllers. Figures 12–16 show that the effect of the external disturbance on the closed-loop system is small, and acceptable in both theoretical and practical work. The convergence time is 23 s and 26 s with the final attitude error $|q_{ei}| < 10^{-2}$ and $|q_{ei}| < 5 \times 10^{-3}$ under controller (8) and controller (23)/(24), respectively. Meanwhile, the actual control torque is within the limited range.

5 Conclusion

The finite-time coordinated attitude control problem for spacecraft formation flying with input saturation has been investigated in this paper. A full-state feedback finite-time control law is first proposed. For further improvement, an output feedback finite-time controller combined with a filter is designed, which only requires the attitude measurements. In both cases, actuator saturation is taken into consideration explicitly. Using the homogeneous system method, a theoretical analysis is given to prove the finite-time stability of the closed-loop system. Finally, numerical simulation results demonstrate the effectiveness of the proposed control
schemes. Future work will focus on extending the existing results to the attitude tracking case, i.e., the constant reference attitude will be replaced by a time-varying reference attitude trajectory.

Acknowledgment
This present work was supported by the National Natural Science Foundation of China (61273175, 61174200), Program for New Century Excellent Talents in University (NCET-11-0801), Heilongjiang Province Science Foundation for Youths (QC2012C024), and Research Fund for Doctoral Program of Higher Education of China (20132302110028). The authors highly appreciate the above financial supports. The authors would also like to thank the reviewers and the editors for their valuable comments and constructive suggestions that helped to improve the paper significantly.

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