Robust variable structure attitude control with $L_2$-gain performance for a flexible spacecraft including input saturation

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Abstract: This paper presents the design of a robust controller, based on variable structure control, for the rotational manoeuvring and vibration reduction of a flexible spacecraft with input saturation. The dynamic equations of motion are formulated as a finite dimensional mathematical model, but accounting for the infinite number of natural vibration modes of the flexible appendages. Based on this model, a variable structure controller is designed for rotational manoeuvres and vibration suppression, and its exponential stability is demonstrated. The synthesis of the control system assumes that only the pitch angle and its derivative are accessible for feedback, and that the flexible modes are not measured. Saturation limits are introduced into the controller design to cope with the actuation limitations, and the stability of the modified control solution is verified. The prescribed robust performance is obtained by ensuring that the $L_2$-gain synthesis, from a torque disturbance to the penalty output, is less than a specified level. Simulation results are presented for the attitude manoeuvring and elastic mode stabilization of an orbiting flexible spacecraft; these results demonstrate the excellent performance of the proposed controller and illustrate its robustness to external disturbances.

Keywords: variable structure control, flexible spacecraft, attitude control, vibration reduction, input saturation

1 INTRODUCTION

Attitude control is the process of re-orienting a rigid body to a desired attitude or orientation and plays an important role in many engineering applications ranging from various aerospace vehicles (for example, aircraft or satellites) to robotic manipulators. The non-linear control problem associated with rigid spacecraft attitude dynamics has been extensively studied and various linear or proportional–derivative types of stabilizing feedback solutions are available in the literature [1–5]. In recent years, there has been an increased interest in the attitude control of spacecraft with flexible appendages (in particular, in their re-orientation with minimal relative elastic vibration) [6–11]. In practice, the high performance of such systems is limited if a spacecraft does not have distributed sensors to measure the deformation of these flexible structures. More significant challenges arise when issues such as modelling uncertainties, external disturbances, structural deformation of the elastic parts, and limited control input amplitude are considered, and especially when these issues occur simultaneously.

Spacecraft dynamical systems are becoming increasingly complex and highly uncertain, and the insensitivity and robustness of variable structure control (VSC) to certain types of disturbances and uncertainties make them attractive for spacecraft control problems [12–15]. In a variable structure system the control law is a discontinuous function of the state variables, and the control law switches when the trajectory crosses a chosen hyperplane (sliding surface) in state space. However, previous
papers [12–15] assumed that the spacecraft are rigid and no flexible modes were considered. Furthermore these VSC methods require full-state feedback but, in practical situations, the measurement of all of the states of a flexible spacecraft might be neither possible nor feasible. Smart materials [16] may be bonded onto the surface of the flexible structures to measure the vibration response, or observers could be used. However, either solution would make the overall system more complex. Since the output is available, output feedback-based VSC can be used to design a controller, and there have been significant research efforts on the output feedback attitude control of flexible spacecraft [17, 18]. Note that these design methods require information on the bounds on the uncertainties and disturbances for the computation of the control gains. In contrast to the VSC methods, non-linear adaptive control methods do not require the bounds, but include an adaptation mechanism to tune the time-varying controller gains. A variety of adaptive spacecraft controllers have been developed [19–21] to meet the performance requirements for spacecraft with unknown and varying properties, although these controllers demand significant real-time computation. Recently, research has focused on the combination of VSC and adaptive control to develop simple and robust spacecraft controllers that work for a wide range of practical systems [22–24].

An implicit assumption inherent in most of the above control strategies is that the control law is implemented without any regard to actuator amplitude saturation constraints. Of course, any electromechanical control actuation device is subject to amplitude constraints leading to saturation non-linearities. As a consequence, actuator saturation arises frequently in practice and can severely degrade the closed-loop system performance, and in some cases can cause instability. Thus the control input limitations must be included in the controller synthesis. Bošković et al. [25, 26], and the references therein, presented extensive results for rigid spacecraft attitude control systems with actuator saturation non-linearities. However, the main focus of these control schemes is the stability analysis and the control performance is not considered explicitly in the control design. A robust tracking control scheme has been developed for a second-order robot dynamic model with $H_\infty$ or $L_2$-gain performance measures to guarantee arbitrary transient performance with arbitrary disturbance attenuation [27, 28]. However, these control approaches have not been applied to flexible robotic systems including actuator saturation. The requirement of full-state feedback has restricted the control methods that may be applied to the flexible robotic systems and to the flexible spacecraft attitude control system.

This paper develops an output feedback-based variable structure control design for the attitude control of flexible spacecraft to obtain minimal relative elastic vibrations subject to actuator saturation, but also considering mismatched uncertainties/disturbances and the manoeuvring performance. The control performance is evaluated by the $L_2$-gain from the torque level disturbance signal to the penalty signal for the attitude angle and angular velocity of the system. The resulting closed-loop system is shown to be uniformly ultimately bounded stable and the effect of external disturbances on the manoeuvring performance can be attenuated to a prescribed level. For the synthesis of the control law only the pitch angle and the pitch rate are measured and the control law is independent of the elastic modes retained in the state variable model. This is important because often the elastic modes are not easy to measure and theoretically the dynamics are infinite in dimension. Simulation results presented in the paper demonstrate that the attitude angle converges to the demanded terminal value and the flexible modes attain the equilibrium point in the presence of actuator saturation, external disturbances, and uncertainties in the model. This paper is organized as follows. The model of the spacecraft is presented in section 2. The novel control scheme along with a proof of stability using Lyapunov analysis is discussed in section 3. Section 4 shows the application of the algorithm for a flexible spacecraft and the performance in terms of attitude manoeuvres and vibration reduction, and compares this performance with that using conventional control methods. Finally remarks and conclusions are given in section 5.

2 MATHEMATICAL MODEL OF A FLEXIBLE SPACECRAFT

The model of the flexible spacecraft under consideration is shown in Fig. 1. $X$, $Y$, and $Z$ are the axes of the inertial frame, and the fixed frame to the hub is represented by the axes $x$, $y$, and $z$. $w(x, t)$ is the flexible deformation at point $x$ with respect to the $Oxy$ frame, $b$ is the radius of the rigid hub, and $l$ is the distance of the tip mass at the end of the appendage from the centre of the hub. The attitude angle $\theta$ denotes the relative motion of the co-
ordinate frames, and $T_h$ is the control torque. Using the extended Hamilton’s principle, the equations of motion for the flexible spacecraft can be written as \cite{21, 22}

$$J_h\ddot{\theta} + \Phi^T\ddot{\eta} = T_h + T_d + f_1(t, \theta, \dot{\theta}, \eta, \ddot{\eta}) \tag{1a}$$

$$\dddot{\eta} + C\ddot{\eta} + K\eta + \Phi\ddot{\eta} = f_2(t, \theta, \dot{\theta}, \eta, \ddot{\eta}) \tag{1b}$$

where $\eta_j$ ($j = 1, 2, \ldots, n$) are the mass normalized vibration modal co-ordinates, $C = \text{diag}\{2\omega_j^2\}$ and $K = \text{diag}\{\omega_j^2\}$ are the modal damping and stiffness coefficients for the appendages respectively, $J_h$ is the moment of inertia with respect to the appendage bending axis, $T_d(t)$ is the external bounded disturbance, and $f_1$ and $f_2$ are the uncertainty and non-linear coupling terms respectively. Thus there are $n + 1$ degrees of freedom in the model, comprising $n$ vibration modal coordinates and the attitude angle of the hub. The model is restricted to elastic transverse bending of the appendages only in the orbital plane \cite{21, 22}. The flexible appendages are assumed to be symmetric, and hence the vibration mode shapes will be either symmetric or anti-symmetric. The symmetric modes are neither controllable nor observable with respect to the attitude angle, and therefore they do not participate in the response and hence do not need to be included in the model, and cannot be controlled using the control torque. The symmetric modes would produce static coupling terms in equation (1), but the anti-symmetric modes will not. The coupling between the rigid and flexible parts is given by the matrix $\Phi$, and the $i$th column is given by $[\Phi_i = \int_0^1 \rho x \phi_i(x) \, dx + m l \dot{\phi}_i(l)]$ where $\phi_i$ is the $i$th mode shape of the appendage, $\rho$ is the mass per unit length of the appendage, and $m$ is the mass located at the end of the appendage.

Viscous damping has been used for convenience because the controller design is developed in the time domain, and structural damping is more suitable for analysis in the frequency domain. However, because a modal representation is used for the dynamics of the appendages, and separate damping ratios are included for each mode, structural damping could be modelled in the time domain using internal variable models, and complex eigenvalues (i.e. a viscous modal damping model) obtained. Indeed damping is difficult to model, and could arise from friction as well as material damping. For spacecraft systems the damping is very small and using a different damping model will have a negligible effect on the simulated controller performance.

Equation (1) can be rewritten in a compact form as

$$Mq + Cq + Kq = [T_h + T_d + f_1 + f_2^T]^T$$

where $q = [\theta, \eta]^T \in \mathbb{R}^{n+1}$, $M = [J_h \quad \Phi^T]$, $C = \text{diag}(0, C)$, $K = \text{diag}(0, K)$, and $E$ is the identity matrix. This system of second-order differential equations in $q$ is then transformed into state-space form to give

$$\dot{x} = Ax + B \text{sat}(u(t)) + f(t, x)$$

\[\text{Fig. 1}\] The single-axis rotation of spacecraft with elastic appendages. On the left is a schematic of the physical system showing the axes fixed to the spacecraft hub, $Oxyz$. On the right is the detailed model of the spacecraft rotating about the $Z$ axis, where the $OXYZ$ axes are fixed in space. The appendages are modelled as beams with flexural rigidity $EI$, mass per unit length $\rho$, tip mass $m$, and elastic displacement $w(x, t)$. The radius of the rigid hub is $b$. 

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There exist two non-negative constants \( k_1 \) and \( k_2 \) such that the sliding motion can be attained.

\[
y = Cx(t)
\]

(3b)

where

\[
x = \begin{bmatrix} q \\ ̇q \end{bmatrix}, \quad A = \begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0_{1 \times n} & 0_{1 \times n} \\ 0 & 0_{1 \times n} & 1 & 0_{1 \times n} \end{bmatrix}
\]

\[
T_h = \text{sat}(u(t)), \quad f(t,x) = \begin{bmatrix} 0 \\ M^{-1} \begin{bmatrix} T_d + f_1 \\ f_2 \end{bmatrix} \end{bmatrix}
\]

Note that the actuator saturation constraint has been included, using the function sat(*) given by

\[
\text{sat}(u) = \tau(t) = \begin{cases} \tau_{\text{max}} & \text{if } u(t) > \tau_{\text{max}} \\ u(t) & \text{if } -\tau_{\text{max}} \leq u(t) \leq \tau_{\text{max}} \\ -\tau_{\text{max}} & \text{if } u(t) < -\tau_{\text{max}} \end{cases}
\]

(4)

where \( \tau_{\text{max}} \) is the maximum amplitude of the control torque. Note that for the synthesis of the controller, only the measured signal vector \( y \), i.e. \( \theta \) and \( \dot{\theta} \), is used for feedback control.

Throughout this paper, the following are assumed.

**Assumption 1.** The triplet \( (A,B,C) \) is controllable and observable.

**Assumption 2.** There exist two non-negative constants \( k_1 \) and \( k_2 \), such that the perturbation \( f(t,x) \) is unknown but bounded by

\[
\|f(x,t)\| \leq k_1 + k_2 \|y(t)\|, \quad \forall (x,t) \in \mathbb{R}^{2(n+1)} \times \mathbb{R}
\]

(5)

**Remark 1**

Assumption 1 is always satisfied for the systems considered. Assumption 2 implies that the unknown perturbation is bounded by a function of the outputs such that the sliding motion can be attained \([29,30]\).

A more general non-linear function could be used to bound the unknown perturbation if this function were bounded in the neighbourhood of the origin \([30]\), and this will be discussed in future work. In addition, for the spacecraft model under consideration, this assumption can be guaranteed in practice from a priori knowledge of the flexible spacecraft system. Fortunately, the values of \( k_1 \) and \( k_2 \) are not required for the controller design.

In this paper, the objective of the control design is to achieve a rest-to-rest manoeuvre in the shortest period of time and vibration reduction in the presence of modelling uncertainties and disturbances subject to control input saturation. Based on the variable structure control technique, the robust controller is then designed so that the dissipation inequality ensuring \( L_2 \)-gain performance is guaranteed to be below a prescribed value, despite torque level disturbance input to the penalty output. More specifically, penalty weighting coefficients, \( \rho_1 > 0 \) and \( \rho_2 > 0 \), are chosen to specify the level of penalty on the output signal, \( z = [\rho_1 \theta \rho_2 \dot{\theta}]^T \), used to evaluate the attitude control performance. Then, for a prescribed level of disturbance attenuation, \( \gamma > 0 \), there exists a control law such that the closed-loop system has the following properties: (a) the equilibrium points in the closed-loop system are uniformly ultimately bounded stable with the input saturation constraint; (b) the torque level disturbance attenuation with respect to the attitude and angular velocity penalty is ensured in the \( L_2 \)-gain sense; (c) the induced elastic vibrations of the flexible appendages during attitude manoeuvring operations are also damped out, i.e. \( \lim_{t \to \infty} \eta = 0 \), \( \lim_{t \to \infty} \dot{\eta} = 0 \); (d) in addition, only the attitude angle and its derivative are measured and thus the elastic modal responses are not available for feedback. A controller for the attitude manoeuvring of a flexible spacecraft is now developed.

### 3 Flexible Spacecraft Control System Design

In the following, the design of a variable structure attitude control system is considered using only the system outputs. Actuator saturation is neglected in section 3.1, and included in section 3.2.

#### 3.1 Variable structure output feedback attitude control

Essential to the design of a variable structure controller is the selection of the switching surface. The control law is then designed such that all of the trajectories are attracted towards this surface and after reaching the surface they slide on it. The structure of the controller changes when the trajectory crosses the switching surface. Here, the linear switching surface, based on the system outputs, is defined as \([29–32]\)

\[
\sigma = G y = G C x = S x
\]

(6)
where $G = [g_1 \ g_2] \in \mathbb{R}^2$ is a constant matrix and $S$ defines the switching surface in state space.

**Remark 2**

The design of the sliding surface based on system outputs given by $G$ follows the procedure given in references [29–32], and only an outline is given here. First the sliding surface in state space, given by $S$, is designed and then the matrix $G$ is selected to satisfy $GC = S$, provided the conditions given by Zak and Hui [32] are met. The matrix $S$ is designed by assigning the complex eigenvalues on the sliding surface, $\{\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}\}$, to prescribed values [29–32]. These prescribed eigenvalues should be stable, so that Re($\lambda_j$) < 0 for $j = 1, 2, \ldots, 2n+1$. To simplify the control design scheme, a state transformation $z = Tx$ is used, where $z$ represents the state variables on the sliding surface. From Assumption 1, there always exist matrices $T \in \mathbb{R}^{2(n+1) \times (2n+1)}$ and $N \in \mathbb{R}^{2(n+1) \times 2(n+1)}$, such that $[A + BN]T = TJ$, where $J \in \mathbb{R}^{2(n+1) \times (2n+1)}$ is the desired Jordan matrix determining the system dynamics restricted to the switching surface. The eigenvalues of $J$, $\lambda_j$, for $j = 1, 2, \ldots, 2n+1$, are the desired eigenvalues in the sliding mode. Let $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ denote the maximum and minimum real parts of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}\}$. The following additional assumption is made.

**Assumption 3**

The matrix $[T \ B]$ is non-singular. The inverse $[T \ B]$ has the form $[T \ B]^{-1} = \begin{bmatrix} \tilde{T} & \tilde{B} \end{bmatrix}$, where $\tilde{T}$ and $\tilde{B}$ denote the generalized inverses of $T$ and $B$, respectively [29–32].

Note that for the spacecraft model considered, Assumption 3 is feasible since rank($B$) = $m$. This assumption is also a necessary condition for the existence of an output feedback sliding mode control that provides a stable sliding motion. The selection $S = \tilde{B}$ is a suitable choice and this is now demonstrated. This sliding surface gives a transformation matrix for the state $x$ as $H = \begin{bmatrix} \tilde{T} \\ S \end{bmatrix}$, with $H^{-1} = [T \ B]$ and

\[
\begin{bmatrix} \dot{z} \\ \dot{\sigma} \end{bmatrix} = Hx. \quad \text{Thus}
\]

\[
\begin{bmatrix} \dot{z} \\ \dot{\sigma} \end{bmatrix} = H(Ax + Bu(t) + f(t, x))
\]

\[
= H \left( AH^{-1} \begin{bmatrix} z \\ \sigma \end{bmatrix} + Bu(t) + f(t, x) \right) \quad (7a)
\]

and the system in equation (3) can be rewritten, using equation (6), as

\[
\dot{z} = f(z) + TF \sigma + Tg(t, x)
\]

\[
\dot{\sigma} = SATz + SAB \sigma + u + Sf(t, x) \quad (7c)
\]

Note that the switching variable $\sigma$ can be measured, whereas the variable $z$ cannot be measured. The following theorem proves that the unmeasured signal $z$ is bounded.

**Theorem 1**

For the system in equation (7b)

\[
\dot{z} = f(z) + TF \sigma + Tg(t, x)
\]

Then the following two statements hold:

(a) $\|\exp(\text{ft})\| \leq k_3 \exp(\lambda_{\text{max}} t)$ for some $k_3 > 0$, and

(b) $\|z(t)\|$ is bounded by

\[
\|z(t)\| \leq k_4 + k_5 \max_{t \leq \tau \leq t} \|\gamma(t)\| \\
\text{where } k_4 \triangleq k_3 \|z(t_0)\| + k_1 \|T\| / \|\lambda_{\text{max}}\|, \quad k_5 \triangleq k_3 (\|TABG\| + \|T\| k_2 / \|\lambda_{\text{max}}\|).
\]

**Proof**

Refer to the Appendix.

For convenience, define

\[
\chi(t, y) \triangleq SATz + Sf(t, x)
\]

Thus equation (7c) can be rewritten as

\[
\dot{\sigma} = SAB \sigma + u + \chi(t, y)
\]

Using equation (9) and Assumption 2, the upper-bound of $\chi(t, y)$ can be obtained as

\[
\|\chi(t, y)\| \leq \|SAT\| \left( k_4 + k_5 \max_{t \leq \tau \leq t} \|\gamma(t)\| \right) + \|S\| (k_1 + k_2 \|\gamma(t)\|)
\]

\[
\leq (k_4 \|SAT\| + k_1 \|S\|) + (k_5 \|SAT\| + k_2 \|S\|) \times \max_{t \leq \tau \leq t} \|\gamma(t)\|
\]

\[
= \gamma_1 + \gamma_2 \max_{t \leq \tau \leq t} \|\gamma(t)\|. \quad (12)
\]

where $\gamma_1 \triangleq (k_4 \|SAT\| + k_1 \|S\|)$, and $\gamma_2 \triangleq (k_5 \|SAT\| + k_2 \|S\|)$. 


Now, the following variable structure output feedback controller is constructed to ensure the occurrence of sliding motion

\[ u(t) = -\text{SAB}\sigma - k_6\sigma \]

\[ -\left( \gamma_1 + \gamma_2 \max_{\tau \in \mathbb{R}} \left( |\theta(\tau)| + \frac{\dot{\theta}(\tau)}{|\sigma|} \right) \right) \left( \frac{\sigma}{|\sigma|} \right) \]  

(13)

where \( k_6 > 0 \). It should be emphasized that this controller uses only the output signals. The controller in equation (13) forces the states of the system in equation (3) to the equilibrium point. The stability of the corresponding control law is given in the following theorem.

**Theorem 2**

Consider the system represented by equation (3) under Assumptions 1–3. If the variable structure output feedback control law is designed as in equation (13) with the sliding surface defined in equation (6) and the parameters \( \gamma_1 \) and \( \gamma_2 \) defined in equation (12), then the switching variable \( \sigma(t) \) converges exponentially to zero, and the states converge to zero (\( x \to 0 \)), as \( t \to \infty \).

**Proof**

Refer to the Appendix.

### 3.2 Variable structure output feedback control with actuator saturation

From a practical perspective, one of the major issues in attitude control system design is that the actuator signal \( u(t) \) in equation (13) generated by the control law might not be realized because of physical constraints [25–27]. A common example of such a constraint is actuator saturation, which imposes a limitation on the magnitude of the achievable control input. Furthermore, the control law in equation (13) requires additional information about the uncertainties and disturbances to determine the boundary values of \( \gamma_1 \) and \( \gamma_2 \) defined in equation (12), since these bounds are important to guarantee the stability of the closed-loop system. In practice, for most dynamic systems, the characteristics of the uncertainties and/or non-linearities of the controlled plant are generally not available or are too expensive and/or difficult to assess. Hence, the boundary values of the perturbations may not be obtained easily, and these difficulties often become a serious practical problem in the application of VSC.

In order to avoid these drawbacks, an alternative controller for flexible spacecraft attitude systems, including actuator saturation, is derived. The actual variable structure output feedback control that is implemented is different to equation (13), and is given as

\[ u(t) = -\text{SAB}\sigma - k_6\sigma - \left( \beta_1 + \frac{\rho_1^2\dot{\theta}^2 + \rho_2^2\dot{\theta}'^2}{|\sigma|} \right) \text{sgn}(\sigma) \]  

(14a)

\[ \text{sat}(u) = \frac{1}{\beta_1}u \]  

(14b)

\[ \beta_1 = \begin{cases} 
\frac{|u|}{\tau_{\text{max}}} & \text{if } |u| > \tau_{\text{max}} \\
1 & \text{otherwise} 
\end{cases} \]  

(14c)

where \( \beta_2 = (\beta_1 - 1) \geq 0 \) and \( \beta_3 \geq \beta_2\tau_{\text{max}} \).

The following theorem guarantees that the system error is still guaranteed to be ultimately bounded, despite the presence of saturation constraints.

**Theorem 3**

Consider the system given in equation (3) under Assumptions 1 and 2 using the control law given by equation (14) under the control parameter constraint

\[ \kappa = k_6 - \frac{1}{4\gamma^2} > 0 \]  

(15)

where \( \gamma \) is a positive constant specifying the attenuation level. Then the closed-loop system satisfies the following:

(a) the equilibrium points can be made uniformly ultimately bounded, i.e. the signals \( \theta, \dot{\theta}, \eta, \) and \( \dot{\eta} \) converge to a ball in the close vicinity of the origin in a stable way;

(b) the \( L_2 \)-gain control performance from the lumped perturbation \( \chi(t, y) \) to \( z = [\rho_1\dot{\theta} \rho_2\dot{\theta}']^T \) is achieved.

**Proof**

Refer to the Appendix.

**Remark 3**

For the controller in equation (14), \( u \) is discontinuous across \( \sigma = 0 \), and this will lead to chattering.
general, chattering must be eliminated for the controller to perform properly. This can be achieved by smoothing out the control discontinuity in a thin boundary layer neighbouring \( \sigma = 0 \). Thus the control input \( u \) is obtained as

\[
  u(t) = -SAB\sigma - k_0 \sigma - \left( \beta_1 + \frac{\rho_1^2 \sigma^2 + \rho_2^2 \sigma^2}{|\sigma| + \varepsilon} \right) \sigma \text{sat}\left( \frac{\sigma}{\varepsilon} \right)
\]

where \( \varepsilon \) is a small value chosen by the designer.

**Remark 4**

In Theorems 1, 2, and 3, there are many parameters to be determined by the controller designer, such as \( \gamma, \rho_1, \rho_2, k_0, \varepsilon \), and \( G \). Note that parameters \( \rho_1 \) and \( \rho_2 \) are the weighting coefficients of the penalty signal \( z \) and are usually selected as unity, while the high robustness to external disturbance or elastic vibrations is guaranteed by prescribing the level \( \gamma \). Theoretically, for a given set of system parameters, a small value for \( \gamma \) can be selected; however, small values of \( \gamma \) will lead to very large control inputs. Since the saturation of the actuators is inevitable in practical problems, a trade-off is required between the choice of a smaller \( \gamma \) and the tracking performance. The optimum choice for \( \gamma \) and \( k_0 \) will only be found by trial-and-error through simulation. The optimum choice for parameters \( \varepsilon \) and \( G \) can be determined easily from the requirements of the closed-loop system.

Based on the analysis and design presented in the previous sections, the design procedure of the proposed control method is summarized as follows.

**Step 1.** Select the desired eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{2n+1} \) with Re\( (\lambda_i) < 0 \), and the associated Jordan matrix \( J \). Then find the corresponding eigenvector matrix \( T \).

**Step 2.** Use the method proposed to find the switching coefficient matrix \( S = \hat{B} \), which also satisfies \( ST = 0 \).

**Step 3.** Check the solvability of the equation \( S = GC \) and design the switching surface function \( \sigma(t) \) in accordance with equation (6) if it is solvable. If it is not solvable, then find another matrix \( S \).

**Step 4.** Use equation (13) with the suitable parameters \( k_0, \rho_1, \rho_2, \gamma, \) and \( \varepsilon \) to design the controller so that the closed-loop system has \( L_2 \)-gain performance as given in section 3.2.

### 3.3 Elastic vibration suppression analysis

When the system is on the linear sliding surface, i.e. \( \sigma = 0 \) and \( \dot{\sigma} = 0 \), then the form of the response is

\[
  g_1 \theta + g_2 \dot{\theta} = 0 \tag{17a}
\]

\[
  g_1 \dot{\theta} + g_2 \ddot{\theta} = 0 \tag{17b}
\]

with \( g_2 \neq 0 \) and \( g_1, g_2 > 0 \). From equation (17a)

\[
  \theta = \theta(t_i)e^{-(g_1/g_2)(t-t_i)} \tag{18}
\]

where \( t_i \) is the time to reach the sliding surface. Substituting equation (13) into equation (1b)

\[
  \tilde{\eta} + C_2 \eta + K_2 \eta + \frac{g_1^2}{g_2^2} e^{-(g_1/g_2)(t-t_i)} \dot{\theta}(t-t_i) \Phi = 0 \tag{19}
\]

where the non-linear term \( f_2(t, \theta, \dot{\theta}, \eta, \dot{\eta}) \) is omitted, for convenience, to enable the analysis of the time response of the elastic vibrations. Equation (19) represents the equation of motion of an underdamped second-order vibration system under the excitation \( \frac{g_1^2}{g_2^2} e^{-(g_1/g_2)(t-t_i)} \dot{\theta}(t-t_i) \). The solution for the \( i \)th modal coordinate is

\[
  \eta_i = l_i e^{-\xi_i \omega_i (t-t_i)} \sin[\omega_i(t-t_i) + \phi_i] + m_i e^{-(g_1/g_2)(t-t_i)} \sin[\omega_i(t-t_i) + \phi_i(t_i)] \tag{20}
\]

where

\[
  \omega_i = \sqrt{\omega_i^2(1-\xi_i^2)} \quad \phi_i = \arctan\left( \frac{\omega_i(1-\xi_i^2)}{-g_1/g_2 + \xi_i \omega_i} \right)
\]

\[
  m_i = \frac{-\Phi_i(g_i^2/g_2^2)}{(g_i^2/g_2^2) - 2 \xi_i \omega_i + \omega_i^2} \tag{21a, b, c}
\]

\[
  l_i = \frac{\Phi_i(g_i^2/g_2^2) \theta(t_i)}{(g_i^2/g_2^2) - 2 \xi_i \omega_i + \omega_i^2} \times \sqrt{\frac{-((g_1/g_2) + \xi_i \omega_i)^2 + \omega_i^2(1-\xi_i^2)}{\omega_i^2(1-\xi_i^2)}} \tag{21d, e}
\]

\[
  \phi_{i0}(t_i) = \arctan\left( \frac{\sqrt{\omega_i^2(1-\xi_i^2)}}{\sqrt{\sqrt{\omega_i^2(1-\xi_i^2)}}} \right)
\]

\[
  \left( \{2 \xi_i \omega_i + \eta_i(t_i)/|\eta_i(t_i)| \} - \xi_i \omega_i \right)
\]
\[ \eta_0(t_t) = \eta(t_t) \times \sqrt{\frac{(2\tilde{\xi}_i\eta_t + \dot{\eta}(t_t))/[\eta_i(t_t)] - \tilde{\xi}_i\eta_t} {\omega_t^2(1 - \tilde{\xi}_i^2)}} \]

From the analysis above, when the system response lies on the sliding surface, the attitude angle and angular velocity decays exponentially, and the vibration motion is a separate externally excited second-order under-damped system. However, when the non-linear term \( f_2(t, \theta, \dot{\theta}, \eta, \dot{\eta}) \) is included, the subsystem in equation (19) is excited by an external bounded disturbance, and the following simulations show the resulting response of the second-order under-damped system. Note that this non-linear term does not alter the stability of the subsystem in equation (19) owing to the boundedness requirement of the disturbance, and in references [17] and [19] the decoupled elastic dynamics were analysed. To this end, the advantage of the variable structure control strategy is that the attitude control and vibration suppression of flexible spacecraft are decoupled.

For comparison, the conventional proportional plus derivative (PD) attitude control for spacecraft of the form \( (T_h = -k_p\theta - k_d\dot{\theta}) \) is employed. The vibration equation is

\[
\dot{\eta} + (E - \Phi f_h^{-1}\Phi^T)^{-1}C_\eta + (E - \Phi f_h^{-1}\Phi^T)^{-1}K_\eta \\
+ (E - \Phi f_h^{-1}\Phi^T)^{-1}\Phi f_h^{-1}(-k_p\theta - k_d\dot{\theta}) = 0
\]

Equation (22) is still a complicated and coupled equation, and the solution cannot be written easily in closed form, as in equation (20), as a separated second-order under-damped system.

### 4 SIMULATION RESULTS

In order to demonstrate the effectiveness of the proposed control scheme, numerical simulations have been performed and are presented in this section. The simulation is performed on the complete spacecraft model given in reference [22]. The first five modal frequencies of the flexible appendage are 3.161, 16.954, 47.233, 94.557, and 153.003 rad/s with all damping ratios equal to 0.004. The first two low-order modes of the five modelled in the flexible system are considered for vibration suppression in this paper. The objective is to slew the spacecraft to a target angle of 60° and the initial conditions are assumed to be \( \theta(0) = 0, \dot{\theta}(0) = 0, \eta(0) = 0, \) and \( \dot{\eta}(0) = 0 \). In addition, the saturation value of the flywheel actuator is 0.5 Nm. In the simulation, four controllers are considered for the purpose of comparison. These cases are:

- (a) attitude manoeuvre control using a proportional–derivative (PD) controller;
- (b) attitude manoeuvre control using the proposed variable structure output feedback controller in equation (13);
- (c) attitude manoeuvre control using the modified variable structure output feedback controller in equation (14);
- (d) attitude manoeuvre control using the variable structure controller with control saturation compensation given in reference [33].

All computations are performed using the Matlab/Simulink software package. The external disturbance used in the simulation is \( T_d(t) = 0.1 \sin (0.01t) + 0.2 \cos (0.02t) + 0.05 \sin (0.012t) \). The simulations also include sensor noise modelled as zero-mean Gaussian random variables. The variances of the sensor noise as \( \sigma_{\eta, s}^2, \sigma_{\dot{\theta}, s}^2, \) and \( \sigma_{\dot{\theta}, p}^2 \), with standard deviations given by:

- (a) gyro scale factor error, \( \sigma_{\eta, s}^2: 0.15 \% (1\sigma) \);
- (b) gyro bias, \( \sigma_{\dot{\theta}, s}^2: 4^\circ /h (1\sigma) \);
- (c) attitude sensor noise, \( \sigma_{\dot{\theta}, p}^2: 0.0002 (1\sigma) \).

The desired closed-loop eigenvalues were selected as \( -0.0171 \pm 16.9531i, -0.0056 \pm 3.1685i, \) and \( -0.0047 \pm 0.0374i \), and this gives the desired response for the PD control case shown in Fig. 2. The vibration energy level, given by \( E = \eta^T K_\eta \), is used to measure the performance of the controllers in the simulations. The control and adaptation gains were selected by trial-and-error until a good performance was obtained for the above cases. The controller parameters of the different methods (the proposed control laws and PD) were determined so that the settling times were similar for all of the controllers.

#### 4.1 PD control

To demonstrate the performance of the attitude control schemes, PD feedback control was applied to control the rigid body motion of the spacecraft. Essentially, the task of the controller design is to manoeuvre the flexible spacecraft to the specified angle of demand. Figures 2(a)–(f) (solid line) show
Attitude manoeuvre control using different control laws. Case 1: PD control (solid line); Case 2: Proposed VSOFC (dashed line); Case 3: VSC with saturation compensation (dotted line); Case 4: proposed VSOFC with $L_2$-gain performance (dash–dot line)

Fig. 2 Attitude manoeuvre control using different control laws. Case 1: PD control (solid line); Case 2: Proposed VSOFC (dashed line); Case 3: VSC with saturation compensation (dotted line); Case 4: proposed VSOFC with $L_2$-gain performance (dash–dot line)
the time responses of the attitude angle, angular velocity, elastic mode displacements, vibration energy of the flexible spacecraft, and flywheel control torque under the step reference input corresponding to a 60° slew. From these responses, the acceptable hub angle and velocity responses were achieved, but
a significant amount of vibration occurred during manoeuvring of the flexible spacecraft as demonstrated in the vibration energy plot of Fig. 2(e) (solid line), which has maximum value of more than 0.0035 Nm. Moreover, the vibration does not settle during slewing. Control saturation is also observed
for this case. Although there still exists some room for improvement using different sets of design parameters, very little improvement in the hub-angle and velocity responses is possible.

4.2 Proposed variable structure output feedback control (VSOFC)

Figures 2(a)–(f) (dashed line) show the results of implementing the proposed variable structure output feedback controller acting on the rigid hub in the presence of disturbances. It is clear from the plots of these responses that the desired angular displacement is accurately achieved by employing the variable structure output feedback control (VSOFC) law. From the comparison between PD control and VSOFC, it can be observed that the relatively large amplitude vibrations excited by rapid manoeuvres can be passively suppressed. Furthermore, it can be seen that the inner-torque of the flywheel approaches zero at 30 s and no saturation phenomenon was observed, even if some small chattering exists.

4.3 Variable structure control with saturation compensation (VSCSC)

For the purpose of further comparison, the system is also controlled by using the variable structure control with saturation compensation (VSCSC) designed in reference [33]. The same simulation case is repeated with the variable structure output feedback control with saturation compensation, and the results are shown in Figs 2(a)–(f) (dotted line). Although the desired angular displacement is achieved and the control saturation is overcome, there exists some small control chattering and some vibration can be observed with the maximum amplitude of vibration energy of more than 0.002 Nm. Thus this controller does not outperform the proposed control law given by equation (14), and further shows the effectiveness of the proposed method for the attitude manoeuvre control and vibration reduction.

4.4 Modified variable structure output feedback control with $L_2$-gain performance

In order to improve the attitude angle response and further reduce the effects of saturation, the modified variable structure output feedback control with $L_2$-gain performance is adopted here and compared with the PD, the proposed VSOFC, and VSCSC cases. Figures 2(a)–(f) (dash-dot line) show the simulation plots for this case under the step input demand. It is noted that an acceptable angular response is achieved and the vibrations have also been significantly reduced in comparison with the PD, VSOFC, and VSCSC controllers as shown in Fig. 2(e) (dash-dot line), so that the maximum amplitude of vibration energy is less than 0.002 Nm. Moreover, no control saturation is observed. This reflects the effectiveness of this modified controller for attitude control and flexible structural vibration suppression.

For the different controllers under two sets of initial conditions, the overall results on settling time, peak vibration energy, peak control torque, and pointing accuracy are summarized in Table 1. The proposed approach not only accomplishes attitude control during manoeuvres, but also simultaneously suppresses the undesired vibrations of the flexible appendages, even though the control saturation and external disturbance are considered explicitly. This control approach also provides a theoretical basis for the practical application of advanced control theory to the flexible spacecraft attitude control system. Note that there may be small variations between samples owing to random bias noise in the simulation results.

Table 1 Performance comparison: settling time, peak vibration energy, peak control torque and pointing accuracy

<table>
<thead>
<tr>
<th>Case</th>
<th>Settling time (s)</th>
<th>Peak vibration energy (Nm)</th>
<th>Peak control torque (Nm)</th>
<th>Pointing accuracy (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD control 1 (To 5% of initial errors)</td>
<td>28</td>
<td>&gt;0.0038</td>
<td>0.5</td>
<td>0.0015</td>
</tr>
<tr>
<td>PD control 2 (To 10% of initial errors)</td>
<td>29</td>
<td>&gt;0.0037</td>
<td>0.5</td>
<td>0.0014</td>
</tr>
<tr>
<td>Proposed VSOFC 1 (To 5% of initial errors)</td>
<td>24</td>
<td>&lt;0.0028</td>
<td>0.5</td>
<td>&lt;0.0005</td>
</tr>
<tr>
<td>Proposed VSOFC 2 (To 10% of initial errors)</td>
<td>24</td>
<td>&lt;0.0028</td>
<td>0.5</td>
<td>&lt;0.0005</td>
</tr>
<tr>
<td>VSOFC with $L_2$-gain 1 (To 5% of initial errors)</td>
<td>24</td>
<td>&lt;0.0002</td>
<td>&lt;0.5</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>VSOFC with $L_2$-gain 2 (To 10% of initial errors)</td>
<td>24</td>
<td>&lt;0.0002</td>
<td>&lt;0.5</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>VSCSC 1 (To 5% of initial errors)</td>
<td>32</td>
<td>&gt;0.002</td>
<td>&lt;0.5</td>
<td>&lt;0.0005</td>
</tr>
<tr>
<td>VSCSC 2 (To 10% of initial errors)</td>
<td>33</td>
<td>&gt;0.003</td>
<td>0.5</td>
<td>&lt;0.0008</td>
</tr>
</tbody>
</table>
inertia matrix of the spacecraft, the simulation was repeated with an uncertainty of \( \Delta J_h \) on the inertia matrix \( J_h \) given by \( \Delta J_h = \pm 0.1 J_h \). The attitude controller parameters were retained, and the responses were very similar to those shown in Figs 2(a)–(f) (dash–dot line), and thus the responses are not shown here. Extensive simulations were also performed with different disturbance inputs. These results show that closed-loop system attitude control and vibration stabilization were accomplished despite perturbations to the system. Moreover, the flexibility in the choice of control parameters can be utilized to obtain a desirable performance while meeting the constraints on the control amplitude and elastic deflection.

5 CONCLUSIONS

In this paper, an approach to attitude manoeuvring control of flexible spacecraft is proposed with unknown disturbances by incorporating a control performance criterion given by an \( L_2 \)-gain constraint in the controller synthesis, subject to the amplitude saturation of actuator. The proposed robust attitude controller with \( L_2 \)-gain performance is based on a variable structure control design, which ensures the global uniform ultimate bounded stability of the system with the \( L_2 \)-gain less than any given small level. Synthesis of the attitude controller using only the attitude and angular rate information feedback is considered here. The elastic vibration is also analysed using the proposed method. Simulations of the slew operation of a spacecraft with a flexible appendage demonstrate that the proposed technique can significantly reduce the vibration of the flexible appendages.

Our future research directions include: (a) the extension of the proposed algorithms to the case of tracking; (b) the combination of these algorithms with active vibration suppression techniques, for example using piezoelectric patches to reduce the vibration further, during and after the manoeuvre operations; (c) the digital implementation of the control scheme on hardware platforms for attitude control experimentation.

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REFERENCES


### APPENDIX

#### Proof of Theorem 1

Since all of the eigenvalues of $J$ have negative real parts, condition (a) holds.

(b) From equation (6), it is easy to conclude

\[
\|TAB\theta + Tf(t, x)\| \leq \|TAB\theta\| + \|Tf(t, x)\| \\
\leq \|TAB\| \|y(t)\| + ||T|| \|f(t, x)\|
\]

Solving equation (8), one obtains

\[
\|z(t)\| \leq e^{\Delta t}\|z(0)\| + \int_0^t e^{(t-\tau)} \|TAB\theta + Tf(t, x)\| \, d\tau \\
\leq k_3 \exp(\lambda_{max}(t-t_0))\|z(t_0)\| \\
+ \max_{t_0 \leq \tau \leq t} \left( \|TAB\theta + Tf(t, x)\| \right) k_3(\exp(\lambda_{max}t) - 1) \lambda_{max} \\
\leq k_3 \|z(t_0)\| + \frac{k_3}{\lambda_{max}^2} \\
\times \max_{t_0 \leq \tau \leq t} \left( \|TAB\| \|y(t)\| + ||T|| \|k_1 + k_2 \|y(t)\| \right) \\
= k_4 + k_5 \max_{t_0 \leq \tau \leq t} \|y(t)\|
\]

This completes the proof.

#### Proof of Theorem 2

To establish the stability of the control scheme, consider the following candidate Lyapunov function $V = \frac{1}{2} \sigma^2$. From equations (3) and (13), the derivative of $V$ with respect to time is

\[
\dot{V} = \sigma \dot{\sigma} = \sigma \left( SAT + SAb\sigma + u + Sf(t, x) \right) \\
\leq \sigma SAb\sigma + \|SAT\| + \|Sf(t, x)\| + au \\
\leq -k_0 \sigma^2 = -2k_0 V
\]
which implies that $V(t) \leq V(0)e^{-2k_0 t}$. Thus, the
switching variable $\sigma$ converges exponentially to zero.
To this end, the definition of the sliding surface in
equation (6) shows that the output state converges to
zero, $y \to 0$, and then $\dot{y} \to 0$ such that the state $x \to 0$.
The proof is completed.

**Proof of Theorem 3**

Using the candidate Lyapunov function $V = \frac{1}{2} \sigma^2$
defined in Theorem 1, the time derivative of this
Lyapunov function can be algebraically rearranged as

$$
\dot{V} = \sigma \dot{\sigma} = \sigma (SAB\sigma + \text{sat}(u) + \chi(t,y))
$$

$$
= \sigma \left(SAB\sigma + \frac{1}{\beta_1} u + \chi(t,y)\right)
$$

$$
= \sigma \left(SAB\sigma - SAB\sigma - k_0 \sigma - \left(\beta_3 + \frac{\rho_1^2 \theta^2 + \rho_2^2 \bar{\theta}^2}{|\sigma|}\right) \text{sgn}(\sigma)\right)
$$

$$
= -k_0 \sigma^2 - \sigma \left(\beta_3 + \frac{\rho_1^2 \theta^2 + \rho_2^2 \bar{\theta}^2}{|\sigma|}\right) \text{sgn}(\sigma)
$$

$$
- \sigma \beta_2 \text{sat}(u) + \sigma \chi(t,y)
$$

$$
\leq -k_0 \sigma^2 - |\sigma| \left(\beta_3 + \frac{\rho_1^2 \theta^2 + \rho_2^2 \bar{\theta}^2}{|\sigma|}\right) + |\sigma| |\text{sat}(u)| \beta_2
$$

$$
+ |\sigma| |\chi(t,y)|
$$

$$
\leq -k_0 \sigma^2 - \left(\frac{\rho_1^2 \theta^2 + \rho_2^2 \bar{\theta}^2}{|\sigma|}\right) + |\sigma| |\chi(t,y)| + |\tilde{z}|^2 - |\tilde{z}|^2
$$

$$
- \gamma^2 |\chi(t,y)|^2 + \gamma^2 |\chi(t,y)|^2
$$

$$
= -k_0 \sigma^2 - \left(1 + \frac{1}{\gamma^2} |\sigma| - \gamma |\chi(t,y)|\right)^2
$$

$$
- |\tilde{z}|^2 + \gamma^2 |\chi(t,y)|^2
$$

$$
+ \frac{1}{4\gamma^2} |\sigma|^2
$$

$$
\leq -k_0 \sigma^2 - |\tilde{z}|^2 + \gamma^2 |\chi(t,y)|^2 + \frac{1}{4\gamma^2} |\sigma|^2
$$

$$
\leq -2kV - |\tilde{z}|^2 + \gamma^2 |\chi(t,y)|^2
$$

(23)

The definition of $V$ and equation (12) gives

$$
\dot{V} \leq -2kV + |\gamma^2 |\chi(t,y)|^2
$$

$$
\leq -2kV + \gamma^2 \left(\gamma_1 + \gamma_2 \max_{0 \leq \tau \leq t}(|\gamma(\tau)|)\right)^2
$$

This implies that the equilibrium points can be
made uniformly ultimately bounded. Integrating the
inequality in equation (23) from $t = 0$ to any $T \geq 0$
with zero initial conditions yields

$$
V(T) + \int_0^T |\tilde{z}|^2 dt \leq \gamma^2 \int_0^T |\chi|^2 dt + V(0)
$$

Thus the $L_2$-gain condition can be guaranteed. The proof for cases (a) and (b) is completed.