Distributed Modal Sensors for Rectangular Plate Structures

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ABSTRACT: Modal sensors are sensitive to the structure’s response at one or more modes and insensitive to the response due to other modes. Design techniques are well established for distributed sensors for beam structures, but plate structures are more difficult unless the sensor thickness or effectiveness is altered. This study considers the design of distributed modal sensors for plate structures. The sensors are assumed to be of constant thickness and the modes of the plate are assumed to be known functions. The boundary is assumed to be smooth and parameterized using a finite number of points and spline interpolation. The approach is demonstrated on a simply supported uniform rectangular plate and the modes are obtained from the analytical solution. The effect of the number of modes of interest, the form of the objective function, and the number of boundary points is highlighted.

Key Words: modal sensors, plate, PVDF film, distributed sensor.

INTRODUCTION

OFTEN it is desirable to excite or measure a single mode of a structure and the idea of using modal sensors and actuators for beam and plate structures has been a subject of intense interest for many years (Lee and Moon, 1990; Irschik, 2002). Using modal sensors in active control reduces the problem of spillover, where high-frequency unmodeled modes can affect the stability of the closed-loop system. Modal sensors reduce or eliminate the response at the higher unmodeled modes and modal actuators ensure that the higher modes are not excited. The sensors and actuators may be either discrete or distributed.

In the design of discrete transducers, the location and gain for each discrete, point transducer must be determined. Established techniques for this are available, based on the mode shape vectors at the actuator or sensor locations (Gawronski, 2000; Friswell, 2001). Distributed transducers are usually based on piezoelectric material and these distributed transducers essentially perform the signal processing by using the charge collecting phenomenon of the material to implement a modal transducer in the spatial domain. Thus distributed piezoelectric transducers alleviate the need for signal processing. Polyvinylidene fluoride (PVDF) film has been widely used for distributed transducers, because electrode pattern shaping is relatively easy and the film is flexible and has low mass. In theory, a continuous gain distribution may be obtained by using the mode shape orthogonality property of the structure. In the case of one-dimensional structures, the gain distribution can be easily implemented by varying the width of the PVDF film along the primary axis of the structure (Lee and Moon, 1990). Friswell (1999) considered modal sensors that cover only part of the beam, and segmented modal sensors for multiple modes. Friswell (2001) used finite element shape functions to design modal sensors and actuators for Euler–Bernoulli beams. Collet (2001) optimized the piezoelectric density to limit observability spill-over and demonstrated the approach on beam structures.

For two-dimensional structures, the required gain distribution can be computed and may be implemented by varying the thickness of the PVDF. However, this is very difficult to achieve in practice. Sun et al. (2002) replaced the actuator layer with variable thickness by many small segments of uniform thickness. Kim et al. (2001, 2002) developed two design methods for distributed modal transducers for composite plates. The first method uses multilayered PVDF films as a single transducer. The electrode pattern, the lamination angle, and the relative poling direction of each PVDF layer are optimized to obtain the desired transducer. In the second method the whole electrode area of a single PVDF film is divided into segments, and the gain imposed on each segment by an interface circuit.
Preumont et al. (2003, 2005) introduced the porous electrode concept, which allows the gains to be introduced by changing the local effectiveness of the electrodes. Sullivan et al. (1997) obtained a similar effect by shading the piezoelectric material. Mukherjee and Joshi (2002a, b) discretized the plate with a fine mesh and within an element the transducer is either present or not. They determined which elements were switched on using an evolutionary algorithm operating at the boundary of the transducer. Kögl and Silva (2005) formulated the problem in a similar way, but used the standard topology optimization approach to determine the sensor shape (Bendsoe and Sigmund, 2003). Hsu and Lee (2004) designed a sensor using even and odd strain functions. Zhang et al. (2005) optimized the size and location of piezoelectric patches, but only considered relatively simple shapes, such as triangles, rectangles, and ellipses.

In this study the authors design a distributed modal transducer for plate structures by optimizing the continuous shape of a constant thickness PVDF film by assuming a smooth boundary. The design method parameterizes the continuous boundary of a sensor and then optimizes the modal output. The approach requires knowledge of the continuous mode shapes, and these are obtained from the analytical solutions for the simply-supported plate used as an example. To the authors’ knowledge this has not been attempted before and has the great advantages of using constant thickness sensor and actuators, and also allowing easier manufacture because of the simple transducer shapes.

SENSOR OUTPUT TO MODAL RESPONSE

Figure 1 shows a plate with a piezoelectric film of uniform thickness \( T \) attached to the upper surface over the area \( \Omega \). If the thickness of the piezoelectric film is negligible compared to the thickness of the plate, then the strain in the piezoelectric film is identical to the strain on the surface of the plate. The total charge generated in the piezoelectric film is (Kim et al., 2001)

\[
q(t) = -\frac{hT}{2} \int_{\Omega} e_{31} \frac{\partial^2 w}{\partial y^2} + e_{32} \frac{\partial^2 w}{\partial y^2} + 2e_{36} \frac{\partial^2 w}{\partial y^2} \, dx \, dy
\]  

(1)

where \( e_{31}, e_{32}, \) and \( e_{36} \) are the piezoelectric strain/charge constants, \( h \) is the thickness of the plate, and \( w \) is the out-of-plane displacement of the plate.

The out-of-plane displacement of the plate, \( w \), can be written as the modal decomposition,

\[
w(x, y, t) = \sum_{i=1}^{\infty} \phi_i(x, y) \eta_i(t)
\]

(2)

where \( \eta_i(t) \) and \( \phi_i(x, y) \) are the \( i \)th modal co-ordinate and mode shape, respectively. It is assumed that the mass and stiffness of the sensor is negligible compared to the properties of the plate. Substituting Equation (2) into Equation (1) yields

\[
q(t) = -\frac{hT}{2} \sum_{i=1}^{\infty} P_i \eta_i(t)
\]

(3)

where

\[
P_i = \int_{\Omega} e_{31} \frac{\partial^2 \phi_i(x, y)}{\partial x^2} + e_{32} \frac{\partial^2 \phi_i(x, y)}{\partial y^2} + 2e_{36} \frac{\partial^2 \phi_i(x, y)}{\partial x \partial y} \, dx \, dy.
\]

(4)

\( P_i \) will be called the modal sensitivity.

PARAMETERIZATION OF THE SENSOR SHAPE

The calculation of the sensitivity of the charge output to the \( i \)th modal co-ordinate, \( P_i \), is more conveniently computed by integrating over the boundary rather than the sensor area. For arbitrary functions \( f(x, y) \) and \( g(x, y) \) Green’s theorem in the plane is

\[
\oint_{\partial \Omega} f(x, y) \, dx + g(x, y) \, dy = \iint_{\Omega} \left( \frac{\partial g(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y} \right) \, dx \, dy,
\]

(5)

where \( \partial \Omega \) is the simple closed boundary curve of the continuous area \( \Omega \).

There are a number of different ways to write the integrand in Equation (4) in terms of \( f(x, y) \) and \( g(x, y) \) so that Green’s theorem may be applied. To retain the symmetry in the co-ordinates \( x \) and \( y \), one can write Equation (4) as

\[
P_i = \oint_{\partial \Omega} \left( e_{31} \frac{\partial \phi_i(x, y)}{\partial x} + e_{32} \frac{\partial \phi_i(x, y)}{\partial y} \right) \, dy - \left( e_{32} \frac{\partial \phi_i(x, y)}{\partial y} + e_{36} \frac{\partial \phi_i(x, y)}{\partial x} \right) \, dx.
\]

(6)
The form of $f(x, y)$ and $g(x, y)$ are immediately apparent by comparing Equations (5) and (6), and the equivalence of Equations (4) and (6) is simple to verify using Equation (5).

Once the boundary is known then the derivative of the boundary curve, $y' = dy/dx$, may be obtained and Equation (6) may be expressed as

$$P_i = \int_{\partial \Omega} f_i(x, y) \, dx$$

where

$$f_i(x, y) = \left( e_{31} \frac{\partial \phi_i(x, y)}{\partial x} + e_{36} \frac{\partial \phi_i(x, y)}{\partial y} \right) y'$$

$$- \left( e_{32} \frac{\partial \phi_i(x, y)}{\partial y} + e_{36} \frac{\partial \phi_i(x, y)}{\partial x} \right).$$

(8)

The closed curve, $\partial \Omega$, will be split into an upper boundary curve, $\partial \Omega_u$, and a lower boundary curve, $\partial \Omega_l$, as shown in Figure 1. The equations of these curves are then

$$y_u = y_u(x) \quad \text{and} \quad y_l = y_l(x).$$

(9)

For optimization purposes the boundaries will be specified by a number of discrete points on these curves $(x_i, y_{iu})$ and $(x_i, y_{il})$. To obtain a smooth curve, spline interpolation is used and the derivatives of the boundary curves may be computed, once the discrete points $(x_i, y_{iu})$ and $(x_i, y_{il})$ are known. Equation (7) may be written as

$$P_i = \int_{x_{min}}^{x_{max}} f_i(x, y_i(x)) \, dx - \int_{x_{min}}^{x_{max}} f_i(x, y_l(x)) \, dx$$

$$= P_i \left( (x_{10}, y_{10}, y_{11}, y_{1l}, y_{1u}, \ldots, y_{N1}, y_{N1}), \right.$$

$$\left. y_{u(N-1)}, x_{max}, y_{N} \right)$$

(10)

and the modal sensitivities may be optimized. Numerical integration methods such as Simpson's Rule may be used to evaluate Equation (10), based on a large number of interior points obtained by the interpolation. A common $x$ position is used for the upper and lower boundary curves in Equation (10), although if desired these positions could be different. In the example the $x$ co-ordinate values will be specified to be equally spaced between the $x_{min}$ and $x_{max}$ values.

In the design of modal sensors for higher modes, the polarization of the PVDF film must be changed. In this case, the sign of the integrand in Equations (1) and (4) must be changed in the re-polled area and there are two different possible scenarios. If the re-polled regions join at a point, then the crossing of the boundary curves $y_u$ and $y_l$ automatically account for the re-poling. Alternatively the regions may have finite length boundaries, which for the lower modes of rectangular plates are straight, and their locations may be obtained from the mode shapes of interest. For more complex structures the boundary between the regions may be parameterized in the same way as the sensor boundaries, and these parameters included in the optimization. Conceptually the most straightforward analysis approach is to split the integral into two parts, consisting of the regions of the sensor with opposite poling. The resulting problem then requires the calculation of the integral along the re-poling line in Equation (6). Note that the boundary of such a sensor will be piecewise differentiable, as required by Equation (8). In practice the re-poling line would have to be a finite width to allow for isolation of the two parts of the sensor, but this is not considered further.

**DESIGN OF DISTRIBUTED MODAL SENSORS**

The design of a modal sensor starts with choosing the number of points to parameterize the boundary curve of the sensor $\partial \Omega$, shown in Figure 1. The positions of these points are the unknown parameters that must be identified and assuming the $x$-values are fixed there will be $2N$ unknowns. The previous section demonstrated how the modal sensitivities, $P_i$, may be obtained as a function of these parameters, given as Equation (10). It remains to specify an objective function and constraints for the optimization procedure. The notion of mode orthogonality will not be used to design the sensor, since this approach is only valid if the effectiveness of the sensor may be varied spatially and the sensor covers the whole plate. Equation (3) showed that a modal sensor for the $k$th mode requires that the sensor output is insensitive to the other modes. Thus,

$$P_i = 0 \text{ for } i \neq k, \quad P_i \neq 0 \text{ for } i = k$$

(11)

With a finite number of parameters it is obviously impossible to achieve a given set of modal sensitivities for an infinite number of modes. The situation is different for the case where the sensor effectiveness may be changed and the orthogonality conditions used to give sensors that are insensitive to the higher modes. In practice only the lower modes of the structure will be of interest and usually low pass filters will be used before any controller is implemented. Controllers such as the positive position feedback controllers have high frequency attenuation built in (Friswell and Inman, 1999). Furthermore, material damping will ensure that the response at the higher modes is attenuated. Thus only the lower $M$ modes will be considered.

Two objective functions will be considered. The first approach is to maximize $J_1$, where

$$J_1 = P_k \left( (x_{min}, y_{0}, y_{10}, y_{1l}, \ldots, y_{0(N-1)}, y_{u(N-1)}, x_{max}, y_{N}) \right)$$

(12)
subject to

\[ P_i = 0, \quad \text{for } i \neq k, \quad 1 \leq i \leq M, \tag{13} \]
\[ 0 \leq x_{\min} < x_{\max} \leq a, \quad y_{\min} \leq y_j \leq y_{\max}, \tag{14} \]
\[ y_{\min} \leq y_j \leq y_{\max}, \quad 0 \leq y_{\min} < y_{\max} \leq b \]

where \( a \) and \( b \) are the length and width of the plate, respectively. Note that it is assumed that the \( x \) co-ordinates of the boundary points are evenly spread between \( x_{\min} \) and \( x_{\max} \), and that boundaries for the sensor width, \( y_{\min} \) and \( y_{\max} \) are specified. Of course it is possible to introduce other cost functions if the response is required to be sensitive to more than one mode.

The second objective function avoids the use of equality constraints by maximizing \( J_2 \) where

\[ J_2 = \frac{P_k^2}{\sum_{i=1}^{M} W_i P_i^2}, \tag{15} \]

subject to the constraints given in Equation (14). The \( W_j \) terms are weights that are used to reduce the output response to the \( i \)th mode shape. It is clear that maximizing \( J_2 \) does not force the sensor output to the response at the unwanted modes to be zero. However, it is possible to add other objectives and one possibility is to make \( J_2 \) not force the sensor output to the response at the unwanted modes only needs to be small, and enforcing a zero constraint may require an impractical solution, or indeed a solution may not be possible. Furthermore, since modeling and manufacturing errors will be present enforcing a zero constraint makes little practical sense. The properties of these two objective functions will be demonstrated in the example.

The solution with a large number of boundary points may not be unique, or only small improvements in the objective function may be obtained. In this situation, it is possible to add other objectives and one possibility is to minimize the curvature of the sensor. Minimizing the curvature will produce a simpler shape and this will potentially make the sensor easier to manufacture. The effect of the curvature needs to be summed along the boundary and a suitable function to include in the objective functions is

\[ C = \int \left( \frac{d^2 y_1}{dx^2} \right)^2 + \left( \frac{d^2 y_2}{dx^2} \right)^2 \, dx. \tag{16} \]

The objective function \( J_3 \) may be modified to give

\[ J_3 = \frac{|P_k|}{P^*} - \frac{C}{C^*} (1 - \alpha) \tag{17} \]

where \( P^* \) and \( C^* \) are chosen such that

\[ 0 < \frac{|P_k|}{P^*} < 1 \quad \text{and} \quad 0 < \frac{C}{C^*} < 1 \tag{18} \]

Varying \( \alpha \) changes the weighting of the modal sensitivity and the curvature in the optimization function, and it is assumed that \( 0 < \alpha < 1 \). In fact \( P^* \) and \( C^* \) are arbitrary, and any redefinition of these constants may be absorbed into the appropriate range of values for \( \alpha \). Methods, such as Pareto sets, exist in the multi-objective optimization literature to choose the optimum value of \( \alpha \) (Steuer, 1986).

**SIMPLY SUPPORTED PLATE MODEL**

Consider a plate of length \( a \) and width \( b \), that is simply supported at all four edges. The natural frequencies and mode shapes of this plate are

\[ \omega_{mn} = \pi^2 \left( \frac{n^2}{a^2} + \frac{n^2}{b^2} \right) \sqrt{\frac{E h^3}{12 \rho (1 - \nu^2)}} \tag{19} \]

\[ \phi_{mn}(x,y) = A_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \tag{20} \]

where \( E \) is the Young’s modulus, \( \nu \) is the Poisson’s ratio, and \( \rho \) is the mass density. For convenience, the natural frequencies are indexed with the integers \( m \) and \( n \) rather than \( i \), although for a given plate the modes would be ordered with increasing natural frequency. \( A_{mn} \) is an arbitrary constant.

Suppose that the sensor’s piezoelectric axis is coincident with the plate axes, \( x \) and \( y \), then \( e_{36} = 0 \). Furthermore, if the sensor is homogeneous then \( e_{31} = e_{32} = e \). Substituting Equation (20) into Equation (6) gives

\[ P_{mn} = e A_{mn} \int_0^a \left( \frac{m \pi x}{a} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right) \, dx \tag{21} \]

Using Equation (10)

\[ P_{mn} = \alpha P_{mn} \hat{P}_{mn} \tag{22} \]

where \( \alpha_{mn} = e A_{mn} \) and

\[ \hat{P}_{mn} = \int_{x_{\min}}^{x_{\max}} \left( \frac{m \pi x}{a} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right) y_j' \left( \frac{n \pi x}{b} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \right) \, dx \tag{23} \]

\( \alpha_{mn} \) is constant for a given mode, and is not related to the shape of the piezoelectric film. \( \hat{P}_{mn} \) does depend on the shape of the piezoelectric film for a given mode, and will be called the modal sensitivity and used in place of \( P_i \) in Equations (12), (13), (15), and (17).

**MODAL SENSORS FOR A SIMPLY SUPPORTED PLATE**

For a plate that is simply supported at all four edges, with length \( a = 200 \) mm and width \( b = 100 \) mm, the first
mode occurs when \( m = n = 1 \) and the second mode is when \( m = 2, \ n = 1 \). The higher modes are ordered according to the natural frequencies determined from Equation (19). The aim will be to design two different modal sensors, one for the first mode and one for the second mode. The example will be used to demonstrate the effect of the number of boundary points, the number of modes of interest, the form of the objective function, and the robustness of sensor shapes. In all cases \( x_{\text{min}} = 0 \) m and \( x_{\text{max}} = 0.2 \) m.

The optimization was performed using the MATLAB function \texttt{fmincon} which uses sequential quadratic programming. The number of iterations necessary depends on the number of boundary points, the modes considered, and the initial sensor shape, although fewer than 100 iterations were usually required. The existence of local maxima was checked by repeating the optimization with a number of random initial sensor shapes. Occasionally a small number of local maxima occurred and in this event the global maximum was chosen from among the local maxima.

**First Mode Sensor**

Suppose that one wishes to design a sensor that is sensitive to the first mode, and insensitive to modes 2–10 \((M = 10)\). Figure 2 shows the results when there are a total of 12 boundary points \((N = 6)\), and therefore 12 unknown \( y \)-co-ordinate values for the boundary points. The equality objective function \( J_1 \), Equation (12), was used and so the modal sensitivities for modes 2–10 are zero. Clearly the design procedure is successful, and the shape is relatively simple. Mode 14 has a high sensitivity, but of course this mode is not involved in the optimization. Reducing the sensitivity of mode 14 will be discussed later. Increasing the number of boundary points makes an insignificant difference to the first 10 modal sensitivities or the sensor shape. Increasing the number of modes of interest up to 13 \((M = 13)\) also has little effect on the sensor design.

**First Mode Sensor – Increasing the Number of Modes**

Suppose the number of modes of interest is now increased above 13 for the equality objective function \( J_1 \), Equation (12). If the number of modes of interest is increased to 14 \((M = 14)\), then no feasible solution is obtained, which means it is impossible to design a modal sensor without re-poling to be sensitive to the first mode but not sensitive to the 14th mode. Note that the 14th mode has \( m = n = 3 \). However it is possible to design a sensor that is insensitive to mode 14 using the second objective function \( J_2 \), Equation (15). Suppose the first 15 modes are of interest \((M = 15)\), and the weights are set to unity \((W_i = 1)\). Figure 3 shows the case when there are 12 boundary points \((N = 6)\). Clearly the first mode dominates, but the sensitivity of the other modes of interest, particular modes 3, 8, and 14, are significant. Increasing the number of boundary points improves the situation and Figures 4–6 show the designs with 16, 18, and 20 boundary points \((N = 8, 9, 10)\). The improved performance is at the expense of more complex sensor shapes. Notice also that some re-poling is required in Figures 4–6.

**First Mode Sensor – Reducing the Sensor Width**

The previous examples assumed that the sensor covered the whole width of the plate. Suppose that the
sensor was constrained so that $y_{\min} = 0.1b$ and $y_{\max} = 0.9b$. Figure 7 shows the results of a sensor sensitive to the first mode using the equality objective function $J_1$, Equation (12), and should be compared to the sensor design given in Figure 2. The number of boundary points is 24 ($N=12$) and 10 modes are considered. The design is clearly successful although the sensor shape is now more complex than that shown in Figure 2.

**Robustness of the Sensor Designs**

Although the sensors may be designed with high precision, errors will occur during manufacture and the model of the structure will contain errors. To demonstrate the robustness of modal sensor shape, small random changes will be made to the $y$-co-ordinates of the optimal discrete points, that is $y_{ij} \rightarrow y_{ij} \pm \Delta$ and
\( y_j \rightarrow y_j \pm \Delta \), for some fixed \( \Delta \), and where the sign is random. The modal sensitivities may then be calculated for this modified sensor. Figures 8 and 9 show two typical cases for the first modal sensor when \( M = 10 \), \( N = 6 \), and \( \Delta = 0.02b \), and these should be compared to Figure 2. Although the sensitivities for modes 2–10 are now non-zero, they are relatively small considering the large value of \( \Delta \) chosen.

Second Mode Sensor

Figure 10 shows the design of a sensor that is sensitive to the second mode, and insensitive to modes 1 and 3–10 (\( M = 10 \)). In this case the polarization of the PVDF film must be changed at \( x = 0.1 \) m. To obtain a solution the number of boundary points has been increased to 16 (\( N = 8 \)). Although the design meets the objectives the sensor shape is complex and asymmetric, and the sensitivity to mode 11 is high. The sensor could be forced to be symmetric by relating the boundary positions on the upper and lower curves. This would also have the advantage of reducing the number of variables in the optimization, although significant a priori knowledge is required. By increasing \( N \), the sensor shape becomes more complex but symmetric, and Figures 11 and 12 show the results for 18 and 24 boundary points (\( N = 9 \) and 12). Increasing the number of modes of interest requires at least 18 boundary points. Figures 13 and 14 show the results for 18 boundary points (\( N = 9 \)) for 11 and 12 modes of interest (\( M = 11 \) or \( 12 \)). Although the design meets the objectives the shape is complex and asymmetric.

Minimizing Curvature

Many of the sensors designed thus far have complex shapes, particularly for the second mode. Minimizing the curvature of the sensor boundary is one option to produce less complex sensor shapes. The objective
function $J_3$, Equation (17), was used to generate a sensor sensitive to the second mode, with $P^* = 1000$ and $C^* = 10$. Figure 15 shows the results for 10 modes of interest ($M = 10$), with 16 boundary points ($N = 8$) and $\alpha = 0.5$. This design should be compared to that of Figure 10, and shows that the sensor shape is much less complex, although the sensitivity to mode 2 is slightly reduced. Increasing the number of boundary points to 24 ($N = 12$), and giving more weight to minimizing curvature by setting $\alpha = 0.2$, makes the sensor boundary even smoother, as shown in Figure 16. Again the penalty is a further slight decrease in the sensitivity to mode 2. Finally, Figure 17 shows the results obtained if more modes are considered ($M = 13$), with a relative small number of boundary points ($N = 9$) and $\alpha = 0.5$.

CONCLUSIONS

This article has presented a method to design distributed modal sensors for plate structures whose mode shapes were obtained analytically. The approach assumes the thickness of the sensor is constant and
optimizes the shape of the sensor boundary. This produces sensor patches with smooth boundaries. Although the study has demonstrated the trade off between performance, in terms of modal sensitivities and the complexity of the sensor shape, extra constraints or performance objectives could be included. Future work will extend the approach to finite element models of structures and the results for the analytical plate models will be used to validate the results from the finite element models. The sensors designed using finite element analysis will also be used for experimental verification since the boundary conditions of the plate could be modeled more easily. The extension to shell structures is also being investigated.

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