Pulsive feedback control of a quarter car model forced by a road profile

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Abstract

We examine the strange chaotic attractor and its unstable periodic orbits for a one degree of freedom nonlinear oscillator with a non-symmetric potential that models a quarter car forced by the road profile. We propose an efficient method of chaos control that stabilizes these orbits using a pulsive feedback technique. A discrete set of pulses is able to transfer the system from one periodic state to another.

In recent years the problem of vehicle vibrations induced by a rough road profile, including their identification and elimination, has received significant interest [1–7]. The application of nonlinear dampers in a vehicle suspension, based on magnetorheological fluid, has opened new perspectives [8–10]. Such solutions have been very effective in eliminating high frequency vibration but have caused new problems because of the nonlinear dynamics. One of these problems is the possibility of a chaotic response to periodic modulation of a road profile. This interesting possibility has already been investigated using a simplified one degree of freedom model, namely a quarter-car model [6,7]. In this paper we examine the geometrical properties of the chaotic solutions [7] and propose an efficient pulsive method to control chaotic vibrations. We use a method introduced recently by Litak et al. [11] to control a nonlinear system characterized by a nonlinear restoring force with a quadratic nonlinearity and linear damping. Here we will prove that this approach may also be successfully applied to the dynamics of quarter-car systems.

The equation of motion of a single degree of freedom quarter-car model (Fig. 1) [6,7] is

\[
m \frac{d^2 x_1}{dt^2} + k_1 (x_1 - x_0) + mg + F_h \left( \frac{dx}{dt} (x_1 - x_0), x_1 - x_0 \right) = 0,
\]

where \(F_h\) is an additional nonlinear hysteretic suspension damping and stiffness force, dependent on relative displacement and velocity, and given by

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The road profile excitation (Fig. 1) is taken as sinusoidal, and is
\[ x_0 = A \sin(\Omega t). \]  
(3)
The system parameters will take the values given by Li et al. [6] as
\[ m = 240 \text{ kg}, \quad k_1 = 160000 \text{ N/m}, \quad k_2 = -300000 \text{ N/m}^3, \quad c_1 = -250 \text{ Ns/m}, \quad c_2 = 25 \text{ Ns}^3/\text{m}^3. \]  
(4)
Introducing the relative coordinate \( x = x_1 - x_0 \) and dimensionless time \( \tau = \omega t \), where \( \omega^2 = k_1/m \) is the linear natural frequency, Eqs. (1) and (2) are transformed to give [7]
\[ \ddot{x} + x + kx^3 + \dot{x} + \beta \dot{x}^3 = -g' + A\Omega^2 \sin(\Omega \tau), \]  
(5)
where \( k = k_2/(m\omega^2), \quad \alpha = c_1/(m\omega), \quad \beta = c_2\omega/m, \quad g' = g/\omega^2 \) and \( \Omega = \Omega'/\omega \). Based on the parameter values given in Eq. (4) the numerical values of these parameters are \( k = 1.875, \quad \alpha = -0.04034, \quad \beta = 2.68957, \quad g' = 0.014715 \).

Following Litak et al. [7] the above equations of motion are simulated to obtain chaotic solutions for some specific values of the amplitude \( A \) and the frequency \( \Omega \). To highlight the bifurcations leading to the chaotic solution, a bifurcation diagram was constructed and the largest Lyapunov exponent calculated as a function of frequency \( \Omega \) for amplitude \( A = 0.41 \). The result is shown in Fig. 2. Both plots have been obtained simultaneously by assuming the initial condition \([x_{in}, \dot{x}_{in}] = [-0.15, 0.1] \). The final position and velocity from the current frequency \( \Omega \) were used as the initial condition for the next \( \Omega \), and the bifurcation diagram has been obtained for both increasing and decreasing frequency.

Fig. 1. The quarter-car model subjected to kinematic excitation with nonlinear damping and stiffness.

\[ F_h \left( \frac{d}{dt}(x_1 - x_0), x_1 - x_0 \right) = k_2(x_1 - x_0)^3 + c_1 \frac{d}{dt}(x_1 - x_0) + c_2 \left( \frac{d}{dt}(x_1 - x_0) \right)^3. \]  
(2)
When $X$ is decreasing (Fig. 2a) one can clearly see several period doubling cascades leading to chaotic vibrations in a small range of $X$ values ($X \in [0.784, 0.804]$). When $X$ is increasing there are several sudden crisis bifurcations separating regular and chaotic regions of the $X$ parameter (Fig. 2a). These bifurcations are consistent with the calculated Lyapunov exponents given in Fig. 2b. The actual changes in system behaviour are investigated further in Fig. 3 where several phase portraits and the corresponding Poincare maps are shown. The variety of solutions presented in Fig. 3 (regular with multiple period, Figs. 3a,c,d, or chaotic, Figs. 3b,e,f) that appear for a small range of $X$, motivate the idea to control the strange attractor.

Starting with a strange chaotic attractor is an unrivaled position for system control since it includes an infinite number of unstable periodic orbits [12–14]. Thus the chaotic attractor can, potentially, be redirected to any of a large number of possible attracting time-periodic motions by making only small time-dependent perturbations of available system parameters. In this paper this perturbation will be provided using the pulsive technique presented in [11].

Suppose, for the purpose of system control, that we begin with the chaotic strange attractor shown in Fig. 3f (for $\Omega = 0.8$ as examined in [7]). The examined equation of motion Eq. (5) is equivalent to the following autonomous system of three first-order differential equations

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - kx^3 - x\dot{x} - \beta x^3 - g' + A\Omega^2 \sin(\Omega z), \\
\dot{z} &= 1,
\end{align*}$$

where $z = \tau$, therefore, whenever one attempts to integrate the system Eq. (2), one must pay attention to the fact that the initial conditions must be such that $z_0 = \tau_0$. 

Fig. 3. Phase portraits (denoted by lines) and Poincare maps (by points) of the strange attractor for $A = 0.41$ and following frequencies $\Omega$: $\Omega = 0.785$ (a), 0.787 (b), 0.791 (c), 0.796848 (d), 0.796849 (e) and 0.8 (f).
Using standard recurrence methods [11] the unstable period 1 orbit embedded within its chaotic attractor may be found numerically. This orbit is denoted
\[ \frac{x}{C_3}(s) = \frac{x}{C_3}(s), \quad \frac{y}{C_3}(s) = \frac{y}{C_3}(s) \]
and is plotted in Fig. 4a. Here, for the integration, a fourth order Runge–Kutta method with a time step equal to \( \frac{2\pi}{1000X} \) was used (i.e. one cycle of period \( T = \frac{2\pi}{\Omega} \) was divided into 1000 equal intervals for the purposes of integration). Using a feedback technique, we have been able to stabilize the unstable period 1 orbit of this system within its chaotic attractor. This may be done by adding a small perturbation
\[ \epsilon \left( X(s) - X^*(s) \right) \]
(\( X^*(s) = [x^*(s), y^*(s)] \))
to the system given by Eq. (6) and \( X(s) = [x(s), y(s)] \).

Following [11] we decided to apply the pulsive feedback technique to the system. Each pulse was set to last for a time equal to the integration step and the interval between two successive pulses was taken as \( \frac{\pi}{3\Omega} \). The system may be driven to different unstable orbits by setting the pulse amplitude, \( \epsilon \), to a given value. We show two examples to illustrate this approach. Using the above procedure with pulse strength \( \epsilon = -0.10 \) we stabilized a period 1 orbit (as depicted in Fig. 4b), while pulse strength \( \epsilon = -0.04 \) stabilized a period 2 orbit (Fig. 4c). Consequently we obtained a period 4 orbit (Fig. 4d) for \( \epsilon = -0.02 \) and a period 8 orbit (Fig. 4e) for \( \epsilon = -0.013 \). In Fig. 4f we plot the corresponding bifurcation diagram for \( x \) versus \( \epsilon \). One can see a period doubling cascade and transition to chaos for a small \( |\epsilon| \). Of course in the limit of \( \epsilon \rightarrow 0 \) we get the chaotic solution examined earlier (Fig. 3f and [7]). It is clear that, in the above method, changing a single parameter, \( \epsilon \), can drive the system to any possible multi-periodic orbit.

Fig. 4. Unstable (period 1: (a)) and stabilized (period 1: (b) for \( \epsilon = -0.1 \), period 2: (c) for \( \epsilon = -0.04 \), period 4: (d) for \( \epsilon = -0.02 \) and period 8: (e) for \( \epsilon = -0.013 \) orbits. Singular points in b–e denote corresponding Poincare maps. (e) is the bifurcation diagram for the controlled system using the pulsive feedback technique versus control parameter \( \epsilon \). The arrows in (f) denote examined cases with \( \epsilon = -0.1 \) (1), \( \epsilon = -0.04 \) (2), \( \epsilon = -0.02 \) (3), \( \epsilon = -0.013 \) (4) and \( \epsilon = 0 \) (5). System parameters used in calculations: \( A = 0.41 \) and \( \Omega = 0.8 \).
We have to add that impulsive methods for dynamical systems control and synchronization are known approaches in the field of chaos [15–21]. However in these treatments one has to find a multi-periodic unstable orbit first, and then stabilize this orbit by a sufficiently strong impulsive signal which drives the system to a periodic motion with the expected properties. Our contribution is different because we are using the same period 1 unstable solution to stabilize any other unstable orbit embedded in the chaotic strange attractor.

In summary, we have successfully demonstrated the application of the pulsive control method to a quarter car model. By changing a single small parameter, $\epsilon$, we were able to choose one of the periodic unstable orbit embedded in the chaotic strange attractor and stabilize it. However the practical problems involved in the implementation of the method on a real system are left to future investigations.

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References