A review of robust optimal design and its application in dynamics

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Abstract

The objective of robust design is to optimise the mean and minimize the variability that results from uncertainty represented by noise factors. The various objective functions and analysis techniques used for the Taguchi based approaches and optimisation methods are reviewed. Most applications of robust design have been concerned with static performance in mechanical engineering and process systems, and applications in structural dynamics are rare. The robust design of a vibration absorber with mass and stiffness uncertainty in the main system is used to demonstrate the robust design approach in dynamics. The results show a significant improvement in performance compared with the conventional solution.

Keywords: Robust design; Stochastic optimization; Taguchi; Vibration absorber

1. Introduction

Successful manufactured products rely on the best possible design and performance. They are usually produced using the tools of engineering design optimisation in order to meet design targets. However, conventional design optimisation may not always satisfy the desired targets due to the significant uncertainty that exists in material and geometrical parameters such as modulus, thickness, density and residual strain, as well as in joints and component assembly. Crucial problems in noise and vibration, caused by the variance in the structural dynamics, are rarely considered in design optimisation. Therefore, ways to minimize the effect of uncertainty on product design and performance are of paramount concern to researchers and practitioners.

The earliest approach to reducing the output variation was to use the Six Sigma Quality strategy [1,2] so that ±6 standard deviations lie between the mean and the nearest specification limit. Six Sigma as a measurement standard in product variation can be traced back to the 1920s when Walter Shewhart showed that three sigma from the mean is the point where a process requires correction. In the early and mid-1980s, Motorola developed this new standard and documented more than $16 billion in savings as a result of the Six Sigma efforts. In the last twenty years, various non-deterministic methods have been developed to deal with design uncertainties. These methods can be classified into two approaches, namely reliability-based methods and robust design based methods. The reliability methods estimate the probability distribution of the system’s response
based on the known probability distributions of the random parameters, and is predominantly used for risk analysis by computing the probability of failure of a system. However, the variation is not minimized in the reliability approaches [3], which concentrate on the rare events at the tails of the probability distribution [4]. Robust design improves the quality of a product by minimizing the effect of the causes of variation without eliminating these causes. The objective is different from the reliability approach, and is to optimise the mean performance and minimise its variation, while maintaining feasibility with probabilistic constraints. This is achieved by optimising the product and process design to make the performance minimally sensitive to the various causes of variation. Hence robust design concentrates on the probability distribution near to the mean values.

In this paper, robust optimal design methods are reviewed extensively and an example of the robust design of a passive vibration absorber is used to demonstrate the application of these techniques to vibration analysis.

2. The concept of robust design

The robust design method is essential to improving engineering productivity, and early work can be traced back to the early 1920s when Fisher and Yates [5] developed the statistical design of experiments (DOE) approach to improve the yield of agricultural crops in England. In the 1950s and early 1960s, Taguchi developed the foundations of robust design to meet the challenge of producing high-quality products. In 1980, he applied his methods in the American telecommunications industry and since then the Taguchi robust design method has been successfully applied to various industrial fields such as electronics, automotive products, photography, and telecommunications [6–8].

The fundamental definition of robust design is described as *A product or process is said to be robust when it is insensitive to the effects of sources of variability, even though the sources themselves have not been eliminated* [9]. In the design process, a number of parameters can affect the quality characteristic or performance of the product. Parameters within the system may be classified as signal factors, noise factors and control factors. Signal factors are parameters that determine the range of configurations to be considered by the robust design. Noise factors are parameters that cannot be controlled by the designer, or are difficult and expensive to control, and constitute the source of variability in the system. Control factors are the specified parameters that the designer has to optimise to give the least sensitivity of the response to the effect of the noise factors. For example, in an automotive body in white, uncertainty may arise in the panel thicknesses given as the noise factors. The geometry is then optimised through control factors describing the panel shape, for the different configurations to be analysed determined by the signal factors, such as different applied loads or the response at different frequencies.

A P-diagram [6] may be used to represent different types of parameters and their relationships. Fig. 1 shows the different types of performance variations, where the large circles denote the target and the response distribution is indicated by the dots and the associated probability density function. The aim of robust design is to make
the system response close to the target with low variations, without eliminating the noise factors in the system, as illustrated in Fig. 1(d).

Suppose that \( \mathbf{y} = f(\mathbf{s}, \mathbf{z}, \mathbf{x}) \) denotes the vector of responses for a particular set of factors, where \( \mathbf{s}, \mathbf{z} \) and \( \mathbf{x} \) are vectors of the signal, noise and control factors. The noise factors are uncertain and generally specified probabilistically. Thus \( \mathbf{z} \), and hence \( f \), are random variables. The range of configurations are specified by a set of signal factors, \( V \), and thus \( \mathbf{s} \in V \). One possible mathematical description of robust design is then

\[
\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \left( \max_{\mathbf{w} \in V} E_x \left[ \| f(\mathbf{s}, \mathbf{z}, \mathbf{x}) - \mathbf{t} \|^2 \right] \right)
\]

subject to the constraints,

\[
g_j(\mathbf{s}, \mathbf{z}, \mathbf{x}) \leq 0 \quad \text{for} \quad j = 1, \ldots, m,
\]

where \( E \) denotes the expected value (over the uncertainty due to \( \mathbf{z} \)) and \( \mathbf{t} \) is the vector of target responses, which may depend on the signal factors, \( \mathbf{s} \). Note that \( \mathbf{x} \) is the parameter vector that is adjusted to obtain the optimal solution, given by Eq. (1) as the solution that gives the smallest worse case response over the range of configurations, or signal factors, in the set \( V \). The key step in the robust design problem is the specification of the objective function, and once this has been done the tools of statistics (such as the analysis of variance) and the design of experiments (such as orthogonal arrays) may be used to calculate the solution.

For convenience, the robust design approaches will be classified into three categories, namely Taguchi, optimisation and stochastic optimisation methods. Further details are given in the following sections. The stochastic optimisation methods include those that propagate the uncertainty through the model and hence use accurate statistics in the optimisation. Also included are optimisation methods that are inherently stochastic, such as the genetic algorithm. These methods are generally time consuming, and the statistical representation may be simplified at the expense of accuracy, for example by considering only a linear perturbation from the nominal values of the parameters. The Taguchi methods take this a step further, using only a limited number of discrete values for the parameters and using the design of experiments methodology to limit the number of times the model has to be run. The response data then requires analysis of variance techniques to determine the sensitivity of the response to the noise and control factors.

3. The Taguchi based methods

Taguchi’s approach to the product design process may be divided into three stages: system design, parameter design, and tolerance design [6]. System design is the conceptual design stage where the system configuration is developed. Parameter design, sometimes called robust design, identifies factors that reduce the system sensitivity to noise, thereby enhancing the system’s robustness. Tolerance design specifies the allowable deviations in the parameter values, loosening tolerances if possible and tightening tolerances if necessary [9].

Taguchi’s objective functions for robust design arise from quality measures using quadratic loss functions. In the extension of this definition to design optimisation, Taguchi suggested the signal-to-noise ratio (SNR), \(-10\log_{10}(\text{MSD})\), as a measure of the mean squared deviation (MSD) in the performance. The use of SNR in system analysis provides a quantitative value for response variation comparison. Maximizing the SNR results in the minimization of the response variation and more robust system performance is obtained. Suppose we have only one response variable, \( y \), and only one configuration of the system (so the signal factor may be neglected). Then for any set of control factors, \( \mathbf{x} \), the noise factors are represented by \( n \) sets of parameters, leading to the \( n \) responses, \( y_i \). Although there are many possible SNR ratios, only two will be considered here.

**The target is best SNR.** This SNR quantifies the deviation of the response from the target, \( t \), and is

\[
\text{SNR} = -10\log_{10}(\text{MSD})
= -10\log_{10} \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - t)^2 \right)
= -10\log_{10} \left( S^2 + \left( \bar{y} - t \right)^2 \right)
\]

where \( S \) is the population standard deviation. Eq. (3) is essentially a sampled version of the general optimisation criteria given in Eq. (1). Note that the second form indicates that the MSD is the summation of population variance and the deviation of the population mean from the target. If the control parameters are chosen such that \( \bar{y} = t \) (the population mean is the target value), then the MSD is just the population variance. If the population standard deviation is related to the mean, then the MSD may also be scaled by the mean to give

\[
\text{SNR} = -10\log_{10}(\text{MSD}) = -10\log_{10} \left( \frac{S^2}{\bar{y}^2} \right)
= 10\log_{10} \left( \frac{\bar{y}^2}{S^2} \right).
\]

**The smaller the better SNR.** This SNR considers the deviation from zero and, as the name suggests, penalises large responses. Thus

\[
\text{SNR} = -10\log_{10}(\text{MSD}) = -10\log_{10} \left( \frac{1}{n} \sum_{i=1}^{n} y_i^2 \right)
\]

This is equivalent to the Target-is-Best SNR, with \( t = 0 \).
The most important task in Taguchi's robust design method is to test the effect of the variability in different experimental factors using statistical tools. The requirement to test multiple factors means that a full factorial experimental design that describes all possible conditions would result in a large number of experiments. Taguchi solved this difficulty by using orthogonal arrays (OA) to represent the range of possible experimental conditions. After conducting the experiments, the data from all experiments are evaluated using the analysis of variance (ANOVA) and the analysis of mean (ANOM) of the SNR, to determine the optimum levels of the design variables. The optimisation process consists of two steps: maximizing the SNR to minimize the sensitivity to the effects of noise, and adjusting the mean response to the target response.

Taguchi's techniques were based on direct experimentation. However, designers often use a computer to simulate the performance of a system instead of actual experiments. Ragsdell and d'Entremont [10] developed a non-linear code that applied Taguchi's concepts to design optimisation. In some cases, the optimal design is the least robust, and designers have to make a trade-off between target performance and robustness. Ideally, one should optimise the expected performance over a range of variations and uncertainties in the noise factors.

Although Taguchi's contributions to the philosophy of robust design are almost unanimously considered to be of fundamental importance, there are certain limitations and inefficiencies associated with his methods. Box and Fung [11] pointed out that the orthogonal array method does not always yield the optimal solution and suggested that non-linear optimisation techniques should be employed when a computer model of the design exists. Better results were achieved on a Wheatstone Bridge circuit design problem used by Taguchi. Montgomery [12] demonstrated that the inner array used for the control factors in the Taguchi's approach and the outer array used for noise factors, is often unnecessary and results in a large number of experiments. Tsui [13] showed that the Taguchi method does not necessarily find an accurate solution for design problems with highly non-linear behaviour. An excellent survey of these controversies was the panel discussion edited by Nair [14].

Taguchi's approach has been extended in a number of ways. D'Errico and Zaino [15] implemented a modification of the Taguchi method using Gaussian–Hermite quadrature integration. Yu and Ishii [16] used the fractional factorial method for systems with significant non-linear effects. Otto and Antonsson [17] addressed robust design optimisation with constraints, using constrained optimisation methods. Ramakrishnan and Rao [18,19] formulated the robust design problem as a non-linear optimization problem with Taguchi's loss function as the objective. They used a Taylor series expansion of the objective function about the mean values of the design variables. Other significant publications in this area include those by Mohandas and Sandgren [20], Sandgren [21], and Belegundu and Zhang [22]. Chang et al. [23] extended Taguchi's parameter design to the notion of conceptual robustness. Lee et al. [24] developed robust design in discrete design space using the Taguchi method. The orthogonal array based on the Taguchi concept was utilized to arrange the discrete variables, and robust solutions for unconstrained optimization problems were found.

4. Optimisation methods

The optimisation procedures aim to minimise the objective functions, such as Eq. (1), directly. The uncertainty in the noise factors means that the system performance is a random variable. One option for the robust optimisation is to minimize both the deviation in the mean value, $|\mu - t|$, and the variance, $\sigma^2$, of the performance function, subject to the constraints, where

$$
\mu_i(x, s) = E_s[f(s, z, x)],
$$

$$
\sigma^2_i(x, s) = E_s\left[\left(f(s, z, x) - \mu_i(x, s)\right)^2\right].
$$

(6)

The quantities of mean and the standard deviation of system performance, for given signal and control factors, may be calculated if the joint probability density function (PDF) of the noise factors is known. For most practical applications these PDFs are unknown, but often it is assumed that all variables have independent normal distributions. In this case the joint PDF becomes a product of the individual PDFs. However, evaluating Eq. (6) is extremely time consuming and computationally expensive and approximations using Taylor's series expansions about the mean noise and control factors, $z$ and $x$, may be used. If only the linear terms are retained in the expansion then the mean and variance of the response are readily computed in terms of the mean and variance of the noise factors.

The constraints must also be satisfied. For a worst case analysis the constraints must be satisfied for all values of control and noise factors, and the constraint in Eq. (2) may be approximated [25] as

$$
g_j(z, x) + \sum_i \frac{\partial g_j}{\partial z_i} \Delta z_i + \sum_i \frac{\partial g_j}{\partial x_i} \Delta x_i \leq 0
$$

(7)

where the dependence on the signal factors has been made implicit. The derivatives are evaluated at $z$ and $x$, and $\Delta z_i$ and $\Delta x_i$ represent the deviations of the elements of $z$ and $x$ from these means. Because of the absolute values this approximation is likely to be very conservative. For a statistical analysis the constraint is not always satisfied, and the probability that the con-
A robust design is one that attempts to optimise both the mean and variance of the performance, and is therefore a multi-objective and non-deterministic problem. Optimisation of the mean often conflicts with minimizing the variance, and a trade-off decision between them is needed to choose the best design. The conventional weighted sum (WS) methods to determine this trade-off have serious drawbacks for the Pareto set generation in multi-objective optimisation problems [34]. The Pareto set [35] is the set of designs for which there is no other design that performs better on all objectives. Using weighted sum methods, it may be impossible to achieve some Pareto solutions and there is no assurance that the best one is selected, even if all of the Pareto points are available. Chen et al. [36] used a combination of multi-objective mathematical programming methods and the principles of decision analysis to address the multi-objective optimisation in robust design. The compromise programming (CP) approach, that is the Tchebycheff or min–max method, replaced the conventional WS method. The advantages of the CP method over the WS approach in locating the efficient multi-objective robust design solution (Pareto points) were illustrated both theoretically and through example problems. Chen et al. [37] made the bi-objective robust design optimisation perspective more powerful by using a physical programming approach [38–40], where each objective was controlled with more flexibility than by using CP. Recently, Messac and Ismail-Yahaya [41] formulated the robust design optimization problem from a fully multi-objective perspective, again using the physical programming method. This method allows the designer to express independent preferences for each design metric, design metric variation, design variable, design variable variation, and parameter variation.

An alternative to CP is the preference aggregation method, and Dai and Mourelatos [42] discussed the fundamental differences of these approaches. The compromise programming approach is a technique for efficient optimisation to recover an entire Pareto frontier, and does not attempt to select one point on that frontier. Preference aggregation, on the other hand, can select the proper trade-off between nominal performance and variability before calculating a Pareto frontier. CP relies on an efficient algorithm to generate an entire Pareto frontier of robust designs, while preference aggregation selects the best trade-off before performing any calculation.

Mattson and Messac [43] gave a brief literature survey on how various robust design optimisation methods handle constraint satisfaction. The focus was the variation in constraints caused by variations in the controlled and uncontrolled parameters. The constraints are considered as two types: equality and inequality constraints. Discussions on inequality constraint satisfaction were given by Balling et al. [44], Parkinson et al. [25], Du
and Chen [45], Lee and Park [46] and Wang and Kodial-yam [47]. Research on equality constraint problems are limited to three approaches; to relax the equality constraint [48,49,51], to satisfy the equality constraint in a probabilistic sense [50,51], or to remove the equality constraint through substitution [52].

5. Stochastic optimization

A slightly different strategy for robust design optimization is based on stochastic optimisation. The stochastic nature of the optimisation arises from incorporating uncertainty into the procedure, either as the parameter uncertainty through the noise factors, or because of the stochastic nature of the optimisation procedure.

The earliest work on stochastic optimization can be traced back to the 1950s [53] and detailed information may be obtained from recent books [54,55]. The objective of stochastic optimization is to minimize the expectation of the sample performance as a function of the design parameters and the randomness in the system. Eggert and Mayne [56] gave an introduction to probabilistic optimisation using successive surrogate probability density functions. Some authors [57–59] employed the concept exploration method for robust design optimisation by creating surrogate global response surface models of computationally expensive simulations. The response surface methodology (RSM) is a set of statistical techniques used to construct an empirical model of the relationship between a response and the levels of some input variables, and to find the optimal responses. Lucas [60] and Myers et al. [61] considered the RSM as an alternative to Taguchi’s robust design method. Monte Carlo simulation generates instances of random variables according to their specified distribution types and characteristics, and although accurate response statistics may be obtained, the computation is expensive and time consuming. Mavris and Bande [62] combined the response model with a Monte Carlo simulation to construct cumulative distribution functions (CDFs) and probability density functions (PDFs) for the objective function and constraints. All these methods depend on the sampling of the statistics, whereby the probabilistic distributions of the stochastic input sets are required. One concern is that the response surface approximations might not generate the accurate sensitivities required for robust design [48]. The stochastic finite element method [63] provides a powerful tool for the analysis of structures with parameter uncertainty. Chakraborty and Dey [64] proposed a stochastic finite element method in the frequency domain for analysis of structural dynamic problems involving uncertain parameters. The uncertain structural parameters are modelled as homogeneous Gaussian stochastic fields and discretised by the local averaging method. Numerical examples were presented to demonstrate the accuracy and efficiency of the proposed method. Chen [65] used Monte Carlo simulation and quadratic programming. Schueller [66] gave a recent review on structural stochastic analysis.

Optimisation approaches that are inherently stochastic include techniques such as simulated annealing, neural networks and evolutionary algorithms (EA) (genetic algorithms, evolutionary programming and evolution strategies (ES)), and these have been applied to multi-objective optimisation problems [67–69]. These techniques do not require the computation of gradients, which is important if the objective function relies on estimating moments of the response random variables. Gupta and Li [70] applied mathematical programming and neural networks to robust design optimisation, and the numerical examples showed that the approach is able to solve highly non-linear design optimisation problems in mechanical and structural design. Sandgren and Cameron [71] used a hybrid combination of a genetic algorithm and non-linear programming for robust design optimization of structures with variations in loading, geometry and material properties. Parkinson [72] employed a genetic algorithm for robust design to directly obtain a global minimum for the variability of a design function by varying the nominal design parameter values. The method proved effective and more efficient than conventional optimisation algorithms. Additional studies are required before these methods are suitable for application to large-scale optimisation problems.

6. Applications in dynamics

The most successful applications of robust design are found in the fields of mechanical design engineering (static performance) and process systems, and there have been few applications to the robustness of dynamic performance. Seki and Ishii [73] applied the robust design concept to the dynamic design of an optical pick-up actuator focusing on shape synthesis using computer models and design of experiments. The response in the first bending and torsion modes were selected as measures of undesirable vibration energy. The objective functions were defined as the signal-to-noise ratios of response frequencies and the sensitivities were derived from the design of experiments using an orthogonal array. Hwang et al. [74] optimised the vibration displacements of an automobile rear view mirror system for robustness, defined by the Taguchi concept.

In this paper the robust design approach is applied to the dynamics of a tuned vibration absorber due to parameter uncertainty, using the optimisation approach through non-linear programming. The objective is to determine the stiffness, mass and damping parameters of the absorber, to minimize the displacements of the main system over a large range of excitation frequencies,
The principle of the vibration absorber was attributed to Frahm [75] who found that a natural frequency of a structure could be split into two frequencies by attaching a small spring–mass system tuned to the same frequency as the structure. Den Hartog and Ormonroyd [76] developed a theoretical analysis of the vibration absorber and showed that a damped vibration absorber could control the vibration over a wide frequency range. Brock [77] and Den Hartog [78] gave criteria for the efficient optimum operation of a tuned vibration absorber, such as the relationship of the frequency and mass ratios between the absorber and main system and the relationship between the damping and mass ratio. Jones et al. [79] successfully designed prototype vibration absorbers for two bridges using these criteria. However, the effect of parameter uncertainty and variations in the dynamic performance have not been considered.

We consider the forced vibration of the two degree of freedom system shown in Fig. 2. The original one degree of freedom system, referred to as the main system, consists of the mass \( m_1 \) and the spring \( k_1 \), and the added the system, referred to as the absorber, consists of the mass \( m_2 \), the spring \( k_2 \) and the damper \( c_2 \).

The equations of motion are

\[
m_1 \ddot{q}_1 + c_2(q_1 - \dot{q}_2) + k_1q_1 + k_2(q_1 - q_2) = f_0 \sin(\Omega t) \\
m_2 \ddot{q}_2 + c_2(q_2 - \dot{q}_1) + k_2(q_2 - q_1) = 0
\]  

(10)

Inman [80] and Smith [81] should be consulted for further details of the modelling and analysis of vibration absorbers. Solving these equations for the steady-state solution, gives the amplitude of vibration of the two masses as

\[
q_{1\text{max}} = f_0 \left( \frac{c_2^2 \Omega^2 + (k_2 - m_2 \Omega^2)^2}{c_2^2 \Omega^2 (k_1 - m_1 \Omega^2 - m_2 \Omega^2)^2 + (k_2 m_2 \Omega^2 - (k_1 - m_1 \Omega^2)(k_2 - m_2 \Omega^2))^2} \right)^{1/2} \\
q_{2\text{max}} = f_0 \left( \frac{c_2^2 \Omega^2 + k_2^2}{c_2^2 \Omega^2 (k_1 - m_1 \Omega^2 - m_2 \Omega^2)^2 + (k_2 m_2 \Omega^2 - (k_1 - m_1 \Omega^2)(k_2 - m_2 \Omega^2))^2} \right)^{1/2} 
\]  

(11)

The equations may be non-dimensionalised using a ‘static’ deflection of main system, defined by

\[
q_{\text{1st}} = \frac{f_0}{k_1}
\]  

(12)

to give the non-dimensional displacement of mass \( m_1 \) as

\[
F(\Omega, m_1, k_1) = \frac{q_{1\text{max}}}{q_{\text{1st}}} = k_1 \left( \frac{c_2^2 \Omega^2 + (k_2 - m_2 \Omega^2)^2}{c_2^2 \Omega^2 (k_1 - m_1 \Omega^2 - m_2 \Omega^2)^2 + (k_2 m_2 \Omega^2 - (k_1 - m_1 \Omega^2)(k_2 - m_2 \Omega^2))^2} \right)^{1/2} \\
+ \left( k_2 m_2 \Omega^2 - (k_1 - m_1 \Omega^2)(k_2 - m_2 \Omega^2) \right)^{1/2} \) \right)^{1/2}
\]  

(13)

Suppose a steel box girder footbridge may be represented by a single degree of freedom system with mass \( m_1 = 17500 \text{ kg} \) and stiffness \( k_1 = 3.0 \text{ MN/m} \). The worst case of dynamic loading is considered to be equivalent to a sinusoidal loading with a constant amplitude of 0.48 \( \text{kN} \) at the natural frequency of the bridge. To simulate the environmental changes, the stiffness \( k_1 \) and the mass \( m_1 \) are allowed to undergo 10% variations \( k_1 \pm \Delta k_1, m_1 \pm \Delta m_1 \), and a wide excitation frequency band \( 1 \leq \Omega \leq 3 \sqrt{\frac{k_1}{m_1}} \) is considered. This ensures that the absorber works well for a wide range of possible excitations. The frequency \( \Omega \) is the signal factor, and the range of this frequency gives the set \( \mathcal{V} \). Using the worst-case formulation, the standard deviations of the mass and stiffness in the main system are \( \sigma_{k_1} = \frac{1}{2} \Delta k_1 \) and \( \sigma_{m_1} = \frac{1}{2} \Delta m_1 \). This standard deviation is calculated for a uniform distribution over the 10% parameter variations and is used for the robust design optimisation. However the original intervals are used in the simulations to demonstrate the effectiveness of the solutions. The mean response function and the standard deviation
of the maximum non-dimensional displacement of mass \( m_1 \) may be calculated using the first-order Taylor approximation as

\[
\mu_t = E \left[ \max_{1 \leq ( \Omega, m_1, k_1 )} ( F( \Omega, m_1, k_1 ) ) \right]
\]

\[
\sigma_t = E \left[ \max_{1 \leq ( \Omega, m_1, k_1 )} \left( \frac{ \partial F( \Omega, m_1, k_1 )}{\partial k_1} \right)^2 \sigma^2_{k_1} + \left( \frac{ \partial F( \Omega, m_1, k_1 )}{\partial m_1} \right)^2 \sigma^2_{m_1} \right]
\]

(14)

Here the expected value is evaluated for the uncertain mass and stiffness parameters \( m_1 \) and \( k_1 \), and \( F \) is defined in Eq. (13). The design variables for the absorber are \( m_2 \), \( k_2 \), and \( c_2 \), with lower and upper bounds given by \( m_2 \in [10, 1750] \), \( c_2 \in [10, 2000] \), \( k_2 \in [100, 10^6] \). Therefore, the robust design of a suitable tuned vibration absorber may be formulated as follows:

Minimize: \( [\mu_t(m_2, c_2, k_2), \sigma_t(m_2, c_2, k_2)] \)

Subject to:\n
\[
| \Delta k_1 | \leq 0.1 k_1, \quad | \Delta m_1 | \leq 0.1 m_1
\]

\( 10 \leq m_2 \leq 1750, \quad 10 \leq c_2 \leq 2000, \quad 100 \leq k_2 \leq 10^6 \)

(15)

The first step of robust design is to seek the ideal design (Utopia point) by a minimisation of \( \mu_t \) and \( \sigma_t \) individually as single objective functions. The ideal design obtained is denoted by \( [\mu_t^*, \sigma_t^*] \). Since there are only two objective functions, \( \mu_t \) and \( \sigma_t \), the two functions are combined into a single objective function, \( G \), by the conventional weighted sum method discussed earlier. The objective function is then

\[
G = \alpha \frac{\mu_t}{\mu_t^*} + (1 - \alpha) \frac{\sigma_t}{\sigma_t^*}
\]

(16)

where the weighting factor \( \alpha \in [0, 1] \) represents the relative importance of the two objectives. \( G \) is minimised for \( \alpha \) between 0 and 1 in steps of 0.1, and Fig. 3 shows the results in the objective space, formed by the normalized values of \( \mu_t \) versus \( \sigma_t \). The trade-off between the mean and the standard deviation can clearly be observed. Points 1 and 11 denote the two utopia points, showing the minimized mean (\( \alpha = 1 \)) and variance (\( \alpha = 0 \)) response optimization, respectively. The other points are the optimized results from different weighting of the mean and variance objective functions. This weighted sum approach has similarities to the L-curve method used in regularization [82]. The 11 optimised solutions for the absorber parameters (mass, stiffness and damping) and the recommended solution from vibration textbooks [80,81], are denoted RD1–RD11 and BS, and are listed in Table 1. To visualise all of these responses over the excitation frequency band of interest, the non-dimensional displacements of mass \( m_1 \) for these different parameter sets are plotted in Fig. 4, where TD denotes the text book solution.

Fig. 3. Efficient solutions of a robust multi-objective optimization.

<table>
<thead>
<tr>
<th>Name</th>
<th>( x )</th>
<th>( m_2 ) (kg)</th>
<th>( k_2 ) (N/m)</th>
<th>( c_2 ) (N s/m)</th>
<th>( m_2 m_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RD1</td>
<td>1</td>
<td>203.72</td>
<td>34509</td>
<td>2000</td>
<td>0.011641</td>
</tr>
<tr>
<td>RD2</td>
<td>0.9</td>
<td>204.05</td>
<td>34576</td>
<td>2000</td>
<td>0.01166</td>
</tr>
<tr>
<td>RD3</td>
<td>0.8</td>
<td>209.79</td>
<td>35532</td>
<td>2000</td>
<td>0.011988</td>
</tr>
<tr>
<td>RD4</td>
<td>0.7</td>
<td>239.12</td>
<td>40421</td>
<td>2000</td>
<td>0.013664</td>
</tr>
<tr>
<td>RD5</td>
<td>0.6</td>
<td>251.27</td>
<td>42421</td>
<td>2000</td>
<td>0.014358</td>
</tr>
<tr>
<td>RD6</td>
<td>0.5</td>
<td>259.71</td>
<td>43817</td>
<td>2000</td>
<td>0.01484</td>
</tr>
<tr>
<td>RD7</td>
<td>0.4</td>
<td>310.37</td>
<td>52188</td>
<td>2000</td>
<td>0.017736</td>
</tr>
<tr>
<td>RD8</td>
<td>0.3</td>
<td>348.83</td>
<td>58483</td>
<td>2000</td>
<td>0.019933</td>
</tr>
<tr>
<td>RD9</td>
<td>0.2</td>
<td>404.44</td>
<td>68483</td>
<td>1999.1</td>
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</tr>
<tr>
<td>RD10</td>
<td>0.1</td>
<td>421.27</td>
<td>70238</td>
<td>2000</td>
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<tr>
<td>RD11</td>
<td>0</td>
<td>510.93</td>
<td>84592</td>
<td>2000</td>
<td>0.029196</td>
</tr>
<tr>
<td>BS (book solution)</td>
<td></td>
<td>175</td>
<td>29393</td>
<td>272.2</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Fig. 4. Robust design results compared with the traditional solution.
Most of the damping values for the robust design cases (except for the RD9 case) are selected as the upper limit (2000 Ns/m). Larger damping values are able to reduce the amplitude of the peak response very effectively, although in practice the amount of damping that can be added is limited. The optimum mass ratio \((m_2/m_1)\) is between 1% and 3%, and the value of the mass for the RD1 case, where only the mean response was minimized, is close to that of the BS case. With the decreased weighting on the mean response and correspondingly increasing weighting on the response variance, the absorber mass is gradually increased, and the response shown in Fig. 4 becomes flatter. Although the peak response is decreased in these cases, the response near the main system natural frequency increases.

To demonstrate the effectiveness of the robust design approach, Monte Carlo simulation [83] is used to evaluate the possible response variations of the main system due to the mass and stiffness uncertainty in the main system, at the different optimum points. In Monte Carlo simulation a random number generator produces samples of the noise factors (in this case \(m_1\) and \(k_1\)), which are then used to calculate the response over the frequency range of interest for a given set of control factors. The random number generator approximates the PDF of the noise factors for a large number of samples. The response variations then give some indication of the robustness of the design given by those particular set of control factors. Four representative cases, namely RD1 \((\alpha = 1)\), RD6 \((\alpha = 0.5)\), RD11 \((\alpha = 0)\) and BS, are investigated, and frequency response variations are plotted in Figs. 5–8, together with the corresponding mean response, shown by the solid line. The amplitudes of the response for the three robust design cases (RD1, RD6, RD11) are much smaller than those from the traditional textbook design (BS). The RD1 case has the lowest minimized mean response and the most sensitive variations.

Fig. 5. Monte Carlo simulation of the response variation due to the uncertainty (BS).

Fig. 6. Monte Carlo simulation of the response variation due to the uncertainty (RD1).

Fig. 7. Monte Carlo simulation of the response variation due to the uncertainty (RD11).

Fig. 8. Monte Carlo simulation of the response variation due to the uncertainty (RD6).
while the RD11 case has the highest mean response and the least sensitive variations. The RD6 case is a compromise between the RD1 and RD11 cases, with a reasonable mean response and insensitive variations due to the uncertainty. The textbook solution, shown in Fig. 5, is most sensitive to the mass and stiffness variations in the main system, although increasing the damping in the absorber would improve this performance.

7. Concluding remarks

The state-of-art approaches to robust design optimisation have been extensively reviewed. It is noted that robust design is a multi-objective and non-deterministic problem. The objective is to optimise the mean and minimise the variability in the performance response that results from uncertainty represented through noise variables. The robust design approaches can generally be classified as statistical-based methods and optimisation methods. Most of the Taguchi based methods use direct experimentation and the objective functions for the optimisation are expressed as the signal to noise ratio (SNR) using the Taguchi method. Using the orthogonal array technique, the analysis of variance and analysis of mean of the SNR are used to evaluate the optimum design variables to ensure that the system performance is insensitive to the effects of noise, and to tune the mean response to the target. The optimisation approaches for robust design are based on non-linear programming methods. The objective functions simultaneously optimise both the mean performance and the variance in performance. A trade-off decision must be made, to choose the best design with the maximum robustness. Recently, novel techniques such as simulated annealing, neural networks and the field of evolutionary algorithms have been applied to solving the resulting multi-objective optimisation problem.

The application of robust design optimisation in structural dynamics is very rare. This paper considers the forced vibration of a two degree of freedom system as an example to illustrate the robust design of a vibration absorber. The objective is to minimize the displacement response of the main system within a wide band of excitation frequencies. The robustness of the response due to uncertainty in the mass and stiffness of the main system was also considered, and the maximum mean displacement response and the variations caused by the mass and stiffness uncertainty were minimized simultaneously. Monte Carlo simulation demonstrated significant improvement in the mean response and variation compared with the traditional solution recommended from vibration textbooks. The results show that robust design methods have great potential for application in structural dynamics to deal with uncertain structures.

Acknowledgments

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