MODEL REDUCTION FOR NON-LINEAR ROTATING MACHINES

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SUMMARY: Model reduction methods for linear undamped static structures are well established and efficient, particularly when a time response analysis is required. In rotating machines the effects of damping, gyroscopic moments, and non-linearities are often very important, and a time simulation is often required. In these cases, particularly for non-linearities, the model reduction methods are more difficult to apply. This paper reviews a selection of candidate reduction methods and applies them to a rotating machine with oil film bearings simulated with a non-linear model. Damping and gyroscopic effects are included. Discussion of the accuracy of the reduced models is based on the corresponding response subspaces. Conclusions are therefore reached concerned the most appropriate model reduction strategy for rotating machines.

KEYWORDS: Model reduction, Non-linear dynamics, Rotordynamics.

1. INTRODUCTION

The simulation of rotating machines is vital to ensure the response to unbalance and other excitation, for design optimisation and condition monitoring. Often rotating machines contain local non-linearities, for example fluid bearings, rotor-stator contact or cracks in shafts. The model of the rotor is assumed to be linear and assembled using finite element modelling [1]. The contribution of the linear model representing the oil film bearings can be included in stiffness and damping matrices, although this does produce a non-symmetric stiffness matrix. For a rotor-bearing system supported by oil film bearings represented by a non-linear model, the equations of motion in the stationary frame are

$$M\ddot{q} + ΩG\dot{q} + Kq = Q_b(q, \dot{q}, t) + Q_{ub}(Ωt) - Q_g$$

where $Q_b$ is a vector of bearing forces including the damping terms, $Q_{ub}$ is a vector of out-of-balance forces and $Q_g$ is a vector of gravitational forces. Note that the stiffness matrix $K$ only describes the stiffness properties of the rotor. $M$ and $G$ are the mass and gyroscopic matrices, and $Ω$ is the rotor spin speed. Thus the rotor is modelled as if it was a free-free rotor and the constraints due to the bearings are accounted for by the forces $Q_b$. The vectors $Q_b$, $Q_{ub}$ and $Q_g$ each contain a large number of zero terms. Equation (1) may be converted to state space form to produce a first order ordinary differential equation and integrated in the time domain.

For machines with non-linear interactions, the analysis often requires a time simulation, which is very computationally intensive, especially when many degrees of freedom are required to model the structure. Therefore the model has to be reduced to ensure efficient computation of the response. Many techniques are available for linear, undamped, non-rotating structures, such as static reduction, dynamic reduction, improved reduced system, modal truncation or component mode synthesis. For non-linear and damped rotating machines the application of such methods is relatively straightforward, but the errors introduced are more difficult to assess. This paper reviews the candidate model reduction strategies, and gives an example of a machine with fluid film bearings. The methods are compared by considering the subspace onto which the response is projected, and are assessed by comparing these subspaces to the principal modes of the response of the full system.
2. MODEL REDUCTION

Model reduction is used to reduce the computational effort required in analysing stationary and rotating systems. Obviously it is impossible to emulate the behaviour of a full system with a reduced model and every reduction transformation sacrifices accuracy for speed in some way. Most of the standard methods used in structural dynamics are based on the undamped model, i.e. just using the mass and stiffness matrices, and assume that both these matrices are symmetric, that the mass matrix is positive definite and the stiffness matrix is positive semi-definite. This ensures that a full set of real modes exist, but these assumptions are violated for the stiffness matrix of an oil film bearing.

All model reduction methods calculate a transformation matrix $T$ between the full set of degrees of freedom $q$ and the reduced vector of co-ordinates $q_r$, so that $q = Tq_r$. The transformation is constrained to be real valued. The equation of motion is then transformed to

$$M_r \ddot{q}_r + G_r \dot{q}_r + K_r q_r = Q_{rb}(q_r, q_r, t) + Q_{rub}(\Omega) - Q_{rg}$$

where the reduced matrices are $M_r = T^\top M T$, $G_r = T^\top G T$ and $K_r = T^\top K T$, and the transformed forces are $Q_{rb} = T^\top Q_b$, $Q_{rub} = T^\top Q_{ub}$ and $Q_{rg} = T^\top Q_g$.

2.1. Static Reduction

One of the oldest and most popular reduction methods is static or Guyan reduction [2], where the inertia and damping terms associated with the discarded degrees of freedom of a structure are neglected. In Guyan reduction, the deflection and force vectors, $q$ and $Q$, and the mass and stiffness matrices, $M$ and $K$, are re-ordered and partitioned into separate quantities relating to master (retained) and slave (discarded) degrees of freedom. Although not strictly necessary, the slaves are chosen from the set of unforced degrees of freedom. The stiffness matrix based on this partitioning and reordering is

$$K = \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix}$$

(3)

where subscripts $m$ and $s$ relate to the master and slave co-ordinates respectively. The reduction transformation is then

$$T = \begin{bmatrix} I & -K_{sm}'K_{ss} \end{bmatrix}$$

(4)

and the reduced co-ordinates are the master co-ordinates. Note that reordering of the degrees of freedom implicit in Eq. (3) means that the rows of $T_r$ must be reordered to obtain $T$. Note that any response generated by the reduced matrices is exact only at zero frequency, hence the name static reduction. As the excitation frequency increases the neglected inertia terms become more significant. The process of choosing the master degrees of freedom may be automated in static reduction by considering the magnitude of the ratio of elastic to inertia terms [3]. The procedure is iterative and a single degree of freedom is eliminated at each iteration. This has the advantage that the inversion of $K_{xx}$ is trivial since it is now a scalar quantity. A further advantage is that after each iteration the inertia and stiffness properties associated with the eliminated degree of freedom are redistributed to the retained degrees of freedom before the next degree of freedom to be removed is chosen.

2.2. Dynamic Reduction

Dynamic reduction [4] is a similar approach to static reduction, where the dynamic stiffness matrix at a given frequency is used in place of the stiffness matrix. For the unbalance response the rotor spin speed would be a good choice for this frequency. The Improved Reduced System (IRS) and the iterated IRS introduce extra terms to compensate for the neglected inertia effects [1], however the increased accuracy of the reduced model is gained at the expense of increased computation.

2.3. Modal Reduction based on the Undamped System

Suppose a subset of the eigenvectors of the undamped full system is to be retained in the reduced model. Let this subset be denoted by $U_r$, where each column represents an eigenvector that is retained. Thus $U_r$ will have many more rows (degrees of freedom) than columns (modes). Using these eigenvectors as the transformation, that is $T = U_r$, produces a reduced model that reproduces the chosen natural frequencies and mode shapes. The reduced degrees of freedom are now modal co-ordinates rather than physical or generalised displacements, but this approach is often useful for time simulations, where the transformation must also be applied to the external generalised forces. The System Equivalent Reduction Expansion Process (SEREP), gives a reduced model that reproduces the chosen eigenvalues and eigenvectors, but is based on physical degrees of freedom [1].
2.4. Principal Component Analysis

Suppose that the response of a structure is measured at \( n \) degrees of freedom and at \( m \) samples. Thus \( \mathbf{q}(t_j) \) represents the measurement of all degrees of freedom at the \( j \)th time instant. This time response history is assembled into a matrix given by \( \mathbf{X} = [\mathbf{q}(t_1) \quad \mathbf{q}(t_2) \quad \ldots \quad \mathbf{q}(t_m)] \). The singular value decomposition (SVD) of the matrix \( \mathbf{X} \) is then calculated as \( \mathbf{X} = \mathbf{USV} \) where \( \mathbf{U} \) and \( \mathbf{V} \) are orthogonal matrices of dimensions \( n \times n \) and \( m \times m \) respectively, and \( \mathbf{S} \) is an \( n \times m \) matrix of singular values where only the diagonal terms are non-zero. The diagonal elements of \( \mathbf{S} \) are decreasing and the corresponding columns of \( \mathbf{U} \) define subspaces that capture an increasing percentage of the response. A transformation matrix can be formed from the columns of \( \mathbf{U} \) corresponding the higher singular values. This decomposition is termed a Principal Component Analysis (PCA) or Proper Orthogonal Decomposition (POD).

2.5. Comparing Response Subspaces

When the model is reduced the simulated response is constrained to the subspace spanned by the columns of the transformation matrix. Note that if two transformations, \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \), span the same subspace, then the reduced matrices obtained from Eq. (2) will yield exactly the same natural frequencies, and the response from the model will be the same. The principle angle, \( \theta \), between the response subspaces, defined by \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \), is calculated from the matrix \( \mathbf{C} = \mathbf{T}_1^T \mathbf{T}_2 \). Since the matrices are orthogonal, the singular values of \( \mathbf{C} \) are less than 1, and hence the principal angles, \( \theta_i \), are defined by \( \cos \theta_i = \sigma_i \), where \( \sigma_i \) is the \( i \)th singular value. The subspace angle [5] is then \( \theta = \max (\theta_i) \).

2.6. Application to Rotating Machines

Most model reduction schemes are based on the undamped model, where the stiffness matrix is symmetric. Damping is usually neglected; for static structures the analysis is usually required because the response is large when the damping is low, and so the undamped modes are a good approximation to the response subspace. However, oil film bearings can give a highly damped response. The modes of the damped system may be used, although non-proportional damping will give complex modes and so some post-processing to obtain a real transformation matrix is required. Non-symmetric stiffness matrices often produce complex modes. Gyroscopic terms couple the rotational degrees of freedom in the machine, which will lead to complex mode shapes.

Non-linearities in the rotating machine model may arise from spatial distributed phenomena such geometric effects in the shaft, or localised effects, such as oil-film bearings or rotor-stator contact. Localised non-linearities are usually small in number and so one approach is to reduce the linear part of the model based on a transformation calculated from a linearised model. The linear model of oil-film bearings should be approximated by a symmetric stiffness matrix to generate the reduction transformation.

3. MACHINE MODEL INCLUDING FLUID FILM BEARINGS

The hydrodynamic bearing, often called the oil or fluid film bearing, is extensively used in large rotating machines because of its high load carrying capacity. It consists of a bearing bush in which the shaft or journal rotates. The bush has an internal diameter that is slightly greater than the diameter of the journal, thereby providing a clearance space between the bush and journal. This clearance is typically between 0.1% and 0.2% of the journal diameter.

Oil is fed into the clearance space through one or more holes or grooves and due to its viscosity and the rotation of the journal it is swept circumferentially to create an oil film between the journal and bush. Most bearings are designed to operate with the ratio between the journal displacement and the radial clearance, i.e. the eccentricity \( e \), of about 0.6 – 0.7. Further details are given by Hamrock [6], Smith [7] and Friswell et al. [1].

3.1. Linear Oil Film Bearing Model

The oil film may be modelled using Reynolds‘ equation where it is assumed that the viscosity and density of the fluid is constant throughout the film. An approximate solution in closed form can be determined for the fluid film for a short bearing, and is used here. The eccentricity is obtained as a solution of

\[
e^8 - 4e^6 + (6 - S^2 \left(16 - \pi^2\right))e^4 - (4 + \pi^2S^2)e^2 + 1 = 0 \tag{5}
\]

where \( S = \frac{D\Omega \eta L^3}{8fc^2} \) is called the modified Sommerfeld number, and is known for a particular speed, load and oil viscosity. \( \eta \) is the oil viscosity, \( \Omega \) is the speed of rotation of the shaft, \( L \), \( D \) and \( c \) are the bearing length, diameter and radial clearance, respectively, and \( f \) is the load on the bearing. The smallest root of Eq. (5) is taken, and is always between 0 and 1. Assuming a vertical load, \( \gamma \) is the angle between this vertical load and the direction of the
displacement of the journal, and is given by
\[ \tan \gamma = \frac{\pi \sqrt{1 - \varepsilon^2}}{4 \varepsilon}. \] (6)

Since short bearings are considered, the linearised stiffness and damping matrices about the steady state shaft position are \(2 \times 2\) matrices and the shaft rotations are unconstrained. These matrices may be written in closed form as
\[
\begin{align*}
K_e &= \frac{f}{c} h_0 \begin{bmatrix}
a_{uu} & a_{uv} \\
a_{vu} & a_{vv}
\end{bmatrix}, \\
C_e &= \frac{f}{c\Omega} h_0 \begin{bmatrix}
b_{uu} & b_{uv} \\
b_{vu} & b_{vv}
\end{bmatrix}
\end{align*}
\] (7)

where
\[
\begin{align*}
a_{uu} &= 4 \left( \pi^2 (1 - \varepsilon^2) + 16 \varepsilon^2 \right), \\
a_{uv} &= \frac{\pi \left( \pi^2 (1 - \varepsilon^2)^2 - 16 \varepsilon^4 \right)}{\varepsilon \sqrt{1 - \varepsilon^2}}, \\
a_{vu} &= -\frac{\pi \left( \pi^2 (1 - \varepsilon^2) (1 + 2 \varepsilon^2) + 32 \varepsilon^3 (1 + \varepsilon^2) \right)}{\varepsilon \sqrt{1 - \varepsilon^2}}, \\
a_{vv} &= 4 \left( \pi^2 (1 + 2 \varepsilon^2) + \frac{32 \varepsilon^3 (1 + \varepsilon^2)}{(1 - \varepsilon^2)} \right), \\
b_{uu} &= \frac{2\pi \sqrt{1 - \varepsilon^2} \left( \pi^2 (1 + 2 \varepsilon^2) - 16 \varepsilon^2 \right)}{\varepsilon}, \\
b_{uv} &= b_{vu} = -8 \left( \pi^2 (1 + 2 \varepsilon^2) - 16 \varepsilon^2 \right), \\
b_{vv} &= \frac{2\pi \left( \pi^2 (1 - \varepsilon^2)^2 + 48 \varepsilon^2 \right)}{\varepsilon \sqrt{1 - \varepsilon^2}}, \\
h_0 &= \frac{1}{\left( \pi^2 (1 - \varepsilon^2) + 16 \varepsilon^2 \right)^{3/2}}.
\end{align*}
\]

Clearly the stiffness matrix is not symmetric and thus hydrodynamic bearings introduce anisotropic supports into the machine model.

3.2. Non-linear Oil Film Bearing Model
A non-linear bearing model for a short, oil-film journal bearing was presented by Adiletta et al. [8]. They assumed laminar and isothermal fluid flow and derived expressions for the fluid film forces in terms of the bearing displacements \(u\) and \(v\) as
\[
\begin{pmatrix}
f_x \\
f_y
\end{pmatrix} = \eta \Omega \frac{D}{2c} \begin{pmatrix}
\frac{D}{L} \\
\frac{L}{D}
\end{pmatrix}^2 \begin{pmatrix}
f_x \\
f_y
\end{pmatrix},
\] (8)

where
\[
\begin{pmatrix}
f_x \\
f_y
\end{pmatrix} = -\frac{c \sqrt{(u\Omega - 2v)^2 + (v\Omega + 2u)^2}}{\Omega (c^2 - u^2 - v^2)} \begin{pmatrix}3V (u/c) - G \sin \alpha - 2c \cos \alpha \\ 3V (v/c) + G \cos \alpha - 2c \sin \alpha\end{pmatrix},
\] (9)

\(u\) and \(v\) are the displacements of the rotor in the \(x\) and \(y\) directions respectively. In Eq. (9),
\[
\alpha = \tan^{-1} \left( \frac{\sqrt{c^2 - u^2 - v^2}}{\sqrt{c^2 - u^2 - v^2}} \right) + \frac{\pi}{2} \text{ sign} \left( \frac{v\Omega + 2u}{u\Omega - 2v} \right) \text{ sign} \left( \frac{v\Omega + 2u}{u\Omega - 2v} \right)
\]
\[
G(u, v, \alpha) = \frac{2c}{\sqrt{c^2 - u^2 - v^2}} \left\{ \pi \right\} \left( \frac{v\cos \alpha - u\sin \alpha}{\sqrt{c^2 - u^2 - v^2}} \right) + \frac{\pi}{2} \text{ sign} \left( \frac{v\Omega + 2u}{u\Omega - 2v} \right) \text{ sign} \left( \frac{v\Omega + 2u}{u\Omega - 2v} \right)
\]
\[
V(u, v, \alpha) = \frac{2c^2 + c (v \cos \alpha + u \sin \alpha) G}{c^2 - u^2 - v^2}
\]
\[
S(u, v, \alpha) = \frac{c (u \cos \alpha + v \sin \alpha)}{c^2 - (u \cos \alpha + v \sin \alpha)^2}
\]

4. THE RESPONSE OF AN EXAMPLE MACHINE

A 1.5 m long shaft has a diameter of 0.05 m. The disks are keyed to the shaft at 0.5 m and 1 m from one end. The left disk is 0.07 m thick with a diameter of 0.28 m; the right disk is 0.07 m thick with a diameter of 0.35 m. For the shaft, \(E = 211 \text{ GN/m}^2\), \(G = 81.2 \text{ GN/m}^2\). There is no internal shaft damping, although grounded dashpots with coefficient 200 Ns/m are included in both directions at both disks. For both the shaft and disks, \(\rho = 7810 \text{ kg/m}^3\). The shaft is supported by identical short fluid film bearings at its ends, so that the bearings contribute no additional stiffness to the rotational degrees of freedom. The oil film bearings have a diameter of 100 mm, a length of 30 mm, a radial clearance of 0.1 mm and the oil film has a viscosity of 0.1 Pa s. The static load at each bearing is obtained
by assuming the rotor is in equilibrium, and thus the left and right bearings support loads of 494N and 556N respectively. The shaft is modelled with 6 finite elements and hence there are 28 degrees of freedom in the full model. Figure 1 shows the example machine schematically.

Figure 1 – The two disk and two bearing machine. The springs represent the oil film bearings.

Figure 2 shows the Campbell diagram for this machine, with the linear model for the linear oil film bearings. The first critical speed occurs at approximately 1000rev/min and half speed whirl is clearly present due to the oil film bearings.

Figure 2 – The Campbell diagram for the example rotor supported by linear hydrodynamic bearings. The dashed line is the 1X forcing line and the dot-dashed line is the 0.5X forcing line.

The unbalance response of the machine close to the first critical speed is simulated; the unbalance is assumed to be 5N mm on the left disk. The chosen speeds are 1000rev/min and 1060rev/min, which show a significant difference in response amplitude. The equations of motion are integrated until the transients have decayed to give the steady state response. Initially the response of the full model (with 28 degrees of freedom) is obtained. Figure 3 shows the orbits of the shaft at both bearing locations, for the linear and non-linear bearing models, for a rotor spin speed of 1000rev/min. The response shows a static deflection due to the load on the bearings, together with an unbalance response with relatively low amplitude. In this case the responses for the linear and non-linear bearing models are very close. Figure 3 also shows the response at a rotor spin speed of 1060rev/min, which is close to the resonance. Now the unbalance response amplitude has increased and there is a significant difference between the linear and non-linear bearing models.

Figure 4 shows the frequency spectra for these simulated responses. The steady state (zero frequency) response has been removed from these plots. As expected the linear response is harmonic and only the rotor spin speed is present in the response. Thus the results orbit is elliptical. In contrast the response for the non-linear bearing is periodic, but also contains responses at the harmonics of the rotor spin speed. The increased amplitude of the fundamental frequency at the high spin speed is clear, and in addition the relative amplitude of the harmonics is higher, suggesting a more non-linear response.

Figure 5 shows the result of a Principal Component Analysis of the responses. Here the singular values are given, which gives a good insight into the dimension of the subspace of the response. For the linear bearing model the response is mainly contained in a subspace of dimension 2, because the third singular value is significantly less than the second. This is reasonable, and the subspace corresponds to the calculated linear frequency response vector; this vector is complex and the real and imaginary parts define the response subspace of dimension 2. For the non-linear bearing model the third and fourth singular values are still significant and hence the number of degrees of freedom in the reduced model will have to be higher in the case of a non-linear bearing model.

Figure 6 shows the orbits for a range of reduced models with 4 degrees of freedom. The linear bearing models are used to calculate the transformations. In this case the Guyan and dynamic reduction methods give the same response because the rotor spin speed is relatively low in this example. The modal reduction transformation requires a symmetric stiffness matrix to obtain real modes; two methods to obtain a symmetric stiffness are used, namely to
retain only the diagonal terms in the bearing stiffness matrix, or to take the average of the bearing stiffness matrix and its transpose. Although some of the reduction methods work reasonably well on the linear bearing model, they all perform poorly for the non-linear bearing model. The final reduction method uses the Principal Component Analysis (PCA), where only two degrees of freedom are required for the linear case, and 4 degrees of freedom are used for the non-linear bearing model. The resulting response is excellent, although of course in practice the response of the full model of the machine would not be known a priori. Table 1 gives the subspace angles for these reduction methods; the exact response subspace is taken from the PCA results, with dimension 2 for the linear response and 4 for the non-linear response.

Figure 7 shows how the response improves when the number of degrees of freedom in the reduced model increases. For this example modal reduction based on the diagonal bearing matrix is used, but the other reduction methods will give similar results. Clearly mode shapes 5 to 8 do not significantly improve the response and the bearing non-linearities excite the higher modes of the linear model. The subspace angles in Table 1 confirm these results.

5. CONCLUSIONS

This paper has summarised various transformations to reduce the number of degrees of freedom of models of rotating machines including non-linearities, gyroscopic effects and damping. The performance of these reduction methods was assessed using the subspace of the steady state unbalance response. For responses where the non-
Figure 5 – The singular values of the machine response for the linear (blue plus) and non-linear (green cross) bearing models.

Table 1 – The subspace angles in degrees between the response subspaces of the reduced system compared to the response subspace obtained by PCA. The modal reduction is based on the linear model, where the bearing stiffness matrix, $K_b$, has been transformed to a symmetric matrix, either by retaining only the diagonal terms, or by averaging the bearing stiffness matrix and its transpose.

<table>
<thead>
<tr>
<th>Reduction Method</th>
<th>DoFs</th>
<th>Linear 1000 rev/min</th>
<th>Linear 1060 rev/min</th>
<th>Non-linear 1000 rev/min</th>
<th>Non-linear 1060 rev/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guyan / Dynamic</td>
<td>4</td>
<td>2.29</td>
<td>2.06</td>
<td>87.22</td>
<td>87.55</td>
</tr>
<tr>
<td>Modal, Diagonal $K_b$</td>
<td>4</td>
<td>1.54</td>
<td>1.34</td>
<td>87.11</td>
<td>87.73</td>
</tr>
<tr>
<td>Modal, Symmetric $K_b$</td>
<td>4</td>
<td>2.61</td>
<td>2.17</td>
<td>87.10</td>
<td>87.76</td>
</tr>
<tr>
<td>Modal, Diagonal $K_b$</td>
<td>8</td>
<td>1.50</td>
<td>1.25</td>
<td>78.02</td>
<td>81.49</td>
</tr>
<tr>
<td>Modal, Diagonal $K_b$</td>
<td>12</td>
<td>1.15</td>
<td>0.95</td>
<td>51.15</td>
<td>56.63</td>
</tr>
</tbody>
</table>

Linearity is significant, the response contains harmonics of the rotor spin speed. Thus higher modes of the linearised model are excited and hence more degrees of freedom are required in the reduced model than would be required for an undamped linear model.

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REFERENCES

Figure 6 – The orbits of the shaft at the right bearing for the linear (blue) and non-linear (green) bearing models for different reduction schemes with 4 degrees of freedom. The responses of the full order system are shown in grey.

Figure 7 – The orbits of the shaft at the right bearing for the linear (blue) and non-linear (green) bearing models for different numbers of reduced degrees of freedom for the modal reduction based on the diagonal bearing stiffness matrix. The responses of the full order system are shown in grey.