Efficient solution of the fuzzy eigenvalue problem in structural dynamics

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Abstract
Many analysis and design problems in engineering and science involve uncertainty to varying degrees. This paper is concerned with the structural vibration problem involving uncertain material or geometric parameters, specified as fuzzy parameters. The requirement is to propagate the parameter uncertainty to the eigenvalues of the structure, specified as fuzzy eigenvalues. The usual approach is to transform the fuzzy problem into several interval eigenvalue problems by using the $\alpha$-cuts method. Solving the interval problem as a generalized interval eigenvalue problem in interval mathematics will produce conservative bounds on the eigenvalues. In this paper, a new efficient perturbation method is proposed for bounds of eigenvalues, which does not need derivatives of the stiffness and mass matrices for each uncertain parameter. Also this method does not require a non-negative decomposition of the stiffness and mass matrix pair. Following a review of methods to solve the interval analysis problem, this paper investigates strategies to efficiently calculate the fuzzy eigenvalues. The paper presents a new increment method, in which the parameter intervals reuse results from previous $\alpha$-cuts. The method is demonstrated on a simple cantilever beam with a pinned support.

1 Introduction

Zadeh [1] presented fuzzy set theory in 1965, and subsequently fuzzy theory was widely used in many fields, such as image identification, information processing and artificial intelligence. The fuzzy finite-element method (FFEM), which is based on fuzzy mathematics, was first proposed in the early 1990s. For static analysis, in 1993, Valliappanhe and Pham [2] presented a fuzzy finite-element method for elastic foundations; however, this approach is only suitable when the elastic modulus and Poisson’s ratio of the foundation are fuzzy. Rao and Sawyer [3] developed an optimization-based method for the numerical solution of linear static fuzzy equation. Lu [4], Liu and Qiu [5] found the solution of the structural fuzzy finite-element equation by employing the theory of resolution in interval numbers and the theorem of fuzzy resolution. Other methods to find the solution for dynamics problems have limitations. In 1990, Wang and Ou [6] used the finite element method to study the random vibration systems using a single-degree-of-freedom system with fuzzy excitation inputs. Chen and Rao [7] investigated the numerical solution for the frequencies and displacements of a fuzzy vibration problem. However, the approach produced distorted response membership functions when the parameter membership functions were non-triangular.

The usual solution approach is to transform the fuzzy problem into several interval eigenvalue problems by using the $\alpha$-cut method. This requires an efficient, accurate and general interval analysis method. Solving the interval problem as a generalized interval eigenvalue problem in interval mathematics will produce conservative bounds on the eigenvalues. Qiu et al. presented the parameter vertex solution theorem [8] and the eigenvalue inclusion principle [9]; however, these methods require the non-negative decomposition of the stiffness and mass matrix pair, which is not necessarily satisfied in practical
applications. Qiu et al. [10] proposed a perturbation based method with reduced computational requirements although the estimation is highly conservative. Xia [11] presented a simplified perturbation method, which is more accurate and much simpler than other methods; however, this method also requires the non-negative decomposition of the stiffness and mass matrix pair.

In this paper, a new efficient perturbation method is proposed to calculate the bounds of the eigenvalues. This method does not need the derivatives of the stiffness and mass matrices for each uncertain parameter, and is suitable for general applications as it does not require the non-negative decomposition of the stiffness and mass matrix pair. Following a review of interval analysis methods, this paper investigates strategies to efficiently calculate the fuzzy eigenvalue problem based on this new interval analysis method. The paper presents a new increment method, in which the parameter intervals reuse interval results from previous \( \alpha \)-cuts. The method is demonstrated on a simple cantilever beam with a pinned support.

## 2 Problem statement

In traditional vibration analysis, the eigenvalue problem of the natural vibration of structure without damping can be expressed as

\[
Ku = \lambda Mu
\]  

(1)

where \( K \) denotes the global stiffness matrix, \( M \) denotes the global mass matrix, \( \lambda = \omega^2 \) represents the eigenvalues of stiffness and mass matrix pair which is denoted by \( < K, M > \) and, \( u \) denotes the eigenvectors corresponding to \( \lambda \).

Generally, the structural stiffness matrix, \( K \), and the mass matrix, \( M \), are functions of the structural parameters (such as material, geometry and mass characteristics) denoted by the vector \( b = (b_j) \), for \( j = 1, 2, \ldots, s \). That is,

\[
K = K(b) = K(b_1, b_2, \ldots, b_s), \quad M = M(b) = M(b_1, b_2, \ldots, b_s)
\]  

(2)

Suppose that the structural parameters \( b = (b_j) \) are fuzzy, and denoted by \( \tilde{b} = (\tilde{b}_j) \) for \( j = 1, 2, \ldots, s \). Then Eq. (1) will become the fuzzy eigenvalue problem

\[
\tilde{K} \tilde{u} = \tilde{\lambda} \tilde{M} \tilde{u}
\]  

(3)

where \( \tilde{K} = K(\tilde{b}) \) denotes the fuzzy global stiffness matrix, \( \tilde{M} = M(\tilde{b}) \) denotes the fuzzy global mass matrix, \( \tilde{\lambda} = \tilde{\omega}^2 \) (\( \tilde{\omega} \) is the structural fuzzy natural frequency) represents the fuzzy eigenvalues of the stiffness and mass pair denoted by \( < \tilde{K}, \tilde{M} > \), and, \( \tilde{u} \) denotes the eigenvectors corresponding to \( \tilde{\lambda} \).

In terms of the fundamental principle of fuzzy mathematics, we can obtain an interval eigenvalue problem by taking \( \alpha \)-cuts of Eq. (3) as follows

\[
K(b^\alpha_j)u_\alpha = \lambda_\alpha M(b^\alpha_j)u_\alpha
\]  

(4)

Thus the fuzzy eigenvalue equation at a particular \( \alpha \)-cut, is equivalent to an interval eigenvalue equation with interval parameter vector \( b^\alpha = [\bar{b}, \tilde{b}]_\alpha \), representing the fuzzy parameter vector, \( \tilde{b} \), at the \( \alpha \) membership level.

Equation (4) represents a generalized interval eigenvalue problem at a given \( \alpha \)-cut. The solution set can be expressed as

\[
\Gamma_\alpha = \{ \lambda : \lambda \in \mathbb{R}^n, \quad K(b)u = \lambda M(b)u, \quad b \in b^\alpha \}
\]  

(5)
Generally, the geometric shape of the set $\Gamma_a$ is complicated, and usually impractical to obtain exactly. Instead, we usually seek the interval vector $\lambda_a^I = [\lambda_a^-, \lambda_a^+]$ which contains $\Gamma_a$ and has the narrowest possible interval components.

In accordance with the theorem of resolution, we can deduce the solution for the fuzzy eigenvalues as

$$\tilde{\lambda} = \bigcup_{\alpha \in [0, 1]} \alpha \lambda_a$$

(6)

To sum up, the key to solve the eigenvalue problem with fuzzy parameters is the transformation between the fuzzy eigenvalue equation and the interval eigenvalue equations. First, the structural fuzzy finite element eigenvalue equation is transformed into interval equations using the $\alpha$-cut method which is based on the extension principle. Then, the interval equation is solved to calculate the interval eigenvalues. The interval solutions are then converted into the fuzzy eigenvalues of the structure, which is based on the theorem of resolution.

3 Eigenvalues of structures with bounded parameters

Generally speaking, there are two types of approach for eigenvalues of structures with bounded parameters. For the first type, Qiu et al. presented the parameter vertex solution theorem [8] and the eigenvalue inclusion principle [9]. However these methods require the non-negative decomposition of the stiffness and mass matrix pair, which often limits the application to practical systems. In the following section, we mainly discuss the second type, namely perturbation based methods, which can be applied more generally.

3.1 Perturbation-based methods

The perturbation theory of structure eigenvalue problems considers small variations in the parameters and eigenvalues about a nominal system given by the central parameters $b_c = (b_{cj})$, where $b_{cj} = \frac{b_j + \bar{b}_j}{2}$.

These central parameters give the central mass and stiffness matrices $M_c = M(b_c)$ and $K_c = K(b_c)$, and the corresponding eigenvalues and eigenvectors, $\lambda_{ci}$ and $u_{ci}$. The first-order perturbation equation are then

$$\lambda_i = \lambda_{ci} + \delta \lambda_i$$

(7)

where

$$\delta \lambda_i \approx u_{ci}^T [\delta K - \lambda_{ci} \delta M] u_{ci} = u_{ci}^T \delta Ku_{ci} - \lambda_{ci} u_{ci}^T \delta Mu_{ci}$$

(8)

$\delta M = M - M_c$ and $\delta K = K - K_c$ are the perturbed mass and stiffness matrices. Often the mass and stiffness matrices will be linear functions of the structural parameter vector, so that

$$\delta K = \sum_{j=1}^s \delta b_j K_j, \quad \delta M = \sum_{j=1}^s \delta b_j M_j$$

(9)

where $\delta b_j = b_j - b_{cj}$. Thus $\delta b_j \in [-\Delta b_j, \Delta b_j]$ where $\Delta b_j = \frac{\bar{b}_j - b_j}{2}$. In other cases Eq. (9) is a good first order approximation to the structural matrices, where the matrices are given by

$$M_j = \frac{\partial M}{\partial b_j} \bigg|_{b_j = b_{cj}}, \quad K_j = \frac{\partial K}{\partial b_j} \bigg|_{b_j = b_{cj}}$$

(10)
Substituting Eq. (9) into Eq. (8), we have

\[ \delta \lambda_i \approx u_{ci}^T \left( \sum_{j=1}^{s} \delta b_j K_j \right) u_{ci} - \lambda_{ci} u_{ci}^T \left( \sum_{j=1}^{s} \delta b_j M_j \right) u_{ci} \]

\[ = \sum_{j=1}^{s} \delta b_j \left( u_{ci}^T K_j u_{ci} \right) - \lambda_{ci} \sum_{j=1}^{s} \delta b_j \left( u_{ci}^T M_j u_{ci} \right) \]

\[ = \sum_{j=1}^{s} \delta b_j \left[ u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right] \quad (11) \]

### 3.1.1 Standard perturbation method

From Eq. (11), we have,

\[ \delta \lambda_i \leq \sum_{j=1}^{s} \left| \delta b_j \right| \left| u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right| \leq \sum_{j=1}^{s} \Delta b_j \left| u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right| \quad (12) \]

\[ \delta \lambda_i \geq -\sum_{j=1}^{s} \Delta b_j \left| u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right| \geq -\sum_{j=1}^{s} \Delta b_j \left| u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right| \quad (13) \]

Define

\[ \Delta \lambda_i = \sum_{j=1}^{s} \Delta b_j \left| u_{ci}^T (K_j - \lambda_{ci} M_j) u_{ci} \right| \quad (14) \]

Then the upper and lower bounds of the eigenvalues, \( \lambda_i, \ i = 1, 2, \ldots, n, \) are

\[ \lambda_i = \lambda_{ci} + \Delta \lambda_i, \quad \hat{\lambda}_i = \lambda_{ci} - \Delta \lambda_i \quad (15) \]

This standard perturbation method doesn’t rely on the non-negative decomposition of the stiffness and mass matrix pair. However the calculation of \( M_j \) and \( K_j \) is required.

### 3.1.2 Simplified perturbation method [11]

Xia [11] presented a simplified perturbation method. If the stiffness and mass matrices could be decomposed into the non-negative form shown in Eq. (9), where \( K_j \) and \( M_j \) are positive definite or positive semi-definite matrices, then \( u_{ci}^T K_j u_{ci} \geq 0 \) and \( u_{ci}^T M_j u_{ci} \geq 0 \), and

\[ \delta \lambda_i \leq \sum_{j=1}^{s} \Delta b_j \left( u_{ci}^T K_j u_{ci} \right) + \lambda_{ci} \sum_{j=1}^{s} \Delta b_j \left( u_{ci}^T M_j u_{ci} \right) \quad (16) \]

\[ \delta \lambda_i \geq -\sum_{j=1}^{s} \Delta b_j \left( u_{ci}^T K_j u_{ci} \right) - \lambda_{ci} \sum_{j=1}^{s} \Delta b_j \left( u_{ci}^T M_j u_{ci} \right) \quad (17) \]

Since

\[ K(\Delta b) = \sum_{j=1}^{s} \Delta b_j K_j, \quad M(\Delta b) = \sum_{j=1}^{s} \Delta b_j M_j \quad (18) \]

Eq. (16) and (17) may be written as
Define

\[ \Delta \lambda_i = u_{i_0}^T K (\Delta b) u_{i_0} + \lambda_{i_0} u_{i_0}^T M (\Delta b) u_{i_0} \]  

(21)

Then the upper and lower bound of the eigenvalues \( \lambda_i \), \( i=1,2,\ldots, n \) are

\[ \bar{\lambda}_i = \lambda_{c_i} + \Delta \lambda_i, \quad \underline{\lambda}_i = \lambda_{c_i} - \Delta \lambda_i \]  

(22)

The benefit of this method are the simple calculation and relative high precision as it avoids interval extension and the over-estimation in the standard perturbation method. However this method requires that the stiffness and mass matrices may be written as a non-negative decomposition which has limitations in real applications.

3.1.3 Vertex-Perturbation method

To avoid the over-estimation in the standard perturbation method and avoid the requirement constraints on the stiffness and mass matrices, we propose a more general approach, called the Vertex-Perturbation method.

From Eq. (11), \( \delta \lambda_i \) is a linear function with respect to the uncertain parameters, \( \delta b_j \), \( j=1,2,\ldots, s \). Geometrically, the linear constraints given by \( \delta b_j \in [-\Delta b_j, \Delta b_j] \) define a convex polyhedron, which is called the feasible region. Since the eigenvalue deviation, \( \delta \lambda_i \), is a linear function of the parameters, the eigenvalues also forms a convex region, and the extreme values are thus attained at a vertex of the polyhedron.

Let the set of vertices of the convex region in parameter space are given by

\[ \Xi_\delta = \left\{ \delta b : \delta b = (\delta b_j) \right\}, \text{ where } \delta b_j = -\Delta b_j \text{ or } \delta b_j = \Delta b_j, \text{ for } j=1,\ldots, s \} \],

(23)

which is equivalent to

\[ \Xi = \left\{ b : b = (b_j) \right\}, \text{ where } b_k = b_j \text{ or } b_k = \bar{b}_j, \text{ for } k=1,\ldots, s \} \]

(24)

This set of vertices has \( 2^s \) elements.

Thus the exact the supremum and the infimum of \( \delta \lambda_i \) in Eq. (11) may be determined as

\[ \overline{\delta \lambda}_i = \max_{\delta b \in \Xi_\delta} \left\{ \sum_{j=1}^{s} \delta b_j \left( u_{c_i j}^T K j u_{c_i} \right) - \lambda_{c_i} \sum_{j=1}^{s} \delta b_j \left( u_{c_i j}^T M j u_{c_i} \right) \right\} \]  

(25)

\[ \underline{\delta \lambda}_i = \min_{\delta b \in \Xi_\delta} \left\{ \sum_{j=1}^{s} \delta b_j \left( u_{c_i j}^T K j u_{c_i} \right) - \lambda_{c_i} \sum_{j=1}^{s} \delta b_j \left( u_{c_i j}^T M j u_{c_i} \right) \right\} \]  

(26)

Since
the exact supremum and the infimum of the first-order deviation of the structural eigenvalue, $\delta \lambda_i$, may be determined by the following expressions, which are equivalent to Eqs. (25) and (26),

$$\overline{\delta \lambda_i} = \max_{b \in \Xi} \left\{ u^T_{ci} \left[ K(b) - \lambda_{ci} M(b) \right] u_{ci} \right\}$$

$$\underline{\delta \lambda_i} = \min_{b \in \Xi} \left\{ u^T_{ci} \left[ K(b) - \lambda_{ci} M(b) \right] u_{ci} \right\}$$

This method gives the exact supremum and the infimum of the first-order deviation of the structural eigenvalue, $\delta \lambda_i$, which does not need derivatives of the stiffness and mass matrices for each uncertain parameter and hence is much more efficient than the standard perturbation method. The calculation requirement is the same as the parameter vertex solution theorem presented by Qiu et al. [8], however it does not require specific decompositions of the stiffness and mass matrices and hence is more general.

### 3.2 Approaches to calculate the eigenvalues of structures with bounded parameters

Table 1 clearly shows the comparison of the advantages and disadvantages of different approaches for eigenvalues of structures with bounded parameters.

### 3.3 Numerical validation

The example considers the stepped bar shown in Fig. 1, where the mass densities and Young’s modulii of the elements are $\rho = 7800 \, \text{kg/m}^3$ and $E = 210 \, \text{GN/ m}^2$. Two cases for the stepped bar with uncertain-but-bounded structural parameters will be discussed to compare the different solution methods.
Table 1: Summary of approaches for the eigenvalue problem with bounded parameters

<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation Required</th>
<th>Accuracy</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter vertex solution [8]</td>
<td>2 eigenvalue solutions</td>
<td>High accuracy</td>
<td>Non-negative decompositions required</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eigenvector inclusion [9]</td>
<td>2 eigenvalue solutions</td>
<td>Depends on the uncertain parameters</td>
<td>Non-negative decompositions required</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard perturbation</td>
<td>1 eigenvalue solution, plus derivatives of $M$ and $K$</td>
<td>High accuracy(second order terms neglected)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vertex-Perturbation</td>
<td>2 eigenvalue solutions</td>
<td>High accuracy(second order terms neglected)</td>
<td></td>
</tr>
</tbody>
</table>

Case 1: In this case, the lengths of the three element are $L=0.4m$. The cross-sectional areas and the moments of inertia of the elements are taken as the uncertain-but-bounded parameters given by

\[
A_1^c = 1.44 \times 10^{-2} \text{ m}^2, \quad \Delta A_1 = 0.014 \times 10^{-2} \text{ m}^2 \\
A_2^c = 1.00 \times 10^{-2} \text{ m}^2, \quad \Delta A_2 = 0.010 \times 10^{-2} \text{ m}^2 \\
A_3^c = 0.64 \times 10^{-2} \text{ m}^2, \quad \Delta A_3 = 0.006 \times 10^{-2} \text{ m}^2 \\
I_1^c = 0.20 \times 10^{-4} \text{ m}^2, \quad \Delta I_1 = 0.0002 \times 10^{-4} \text{ m}^2 \\
I_2^c = 0.10 \times 10^{-4} \text{ m}^2, \quad \Delta I_2 = 0.0001 \times 10^{-4} \text{ m}^2 \\
I_3^c = 0.05 \times 10^{-4} \text{ m}^2, \quad \Delta I_3 = 0.00005 \times 10^{-4} \text{ m}^2
\]

The results by different methods are shown in Table 2. The results show that, in this case, the Vertex-Perturbation method obtained the same bounds as the other perturbation methods and narrower bounds than the parameter vertex solution theorem.
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<table>
<thead>
<tr>
<th>Parameter vertex solution theorem</th>
<th>Central value</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 \times 10^5 )</td>
<td>3.651</td>
<td>3.612</td>
<td>3.689</td>
<td>3.612</td>
<td>3.689</td>
<td>3.612</td>
<td>3.689</td>
<td>3.613</td>
<td>3.690</td>
</tr>
<tr>
<td>( \lambda_2 \times 10^6 )</td>
<td>7.336</td>
<td>7.257</td>
<td>7.414</td>
<td>7.257</td>
<td>7.414</td>
<td>7.257</td>
<td>7.414</td>
<td>7.257</td>
<td>7.415</td>
</tr>
<tr>
<td>( \lambda_4 \times 10^8 )</td>
<td>2.581</td>
<td>2.553</td>
<td>2.608</td>
<td>2.553</td>
<td>2.608</td>
<td>2.553</td>
<td>2.608</td>
<td>2.553</td>
<td>2.608</td>
</tr>
<tr>
<td>( \lambda_6 \times 10^9 )</td>
<td>2.740</td>
<td>2.711</td>
<td>2.769</td>
<td>2.711</td>
<td>2.769</td>
<td>2.711</td>
<td>2.769</td>
<td>2.711</td>
<td>2.769</td>
</tr>
</tbody>
</table>

Table 2: Comparison of eigenvalue intervals for the stepped bar example, case 1

Case 2: In this case, the cross-sectional areas and the moments of inertia of the elements are defined to be the same as for case 1. In addition, the lengths of the three elements are also uncertain with central value \( L_c = 0.4m \) and \( \Delta L = 0.008 m \). Table 3 shows that the Vertex-Perturbation method gives similar bounds to the standard perturbation method. However, as the Vertex-Perturbation method is much more efficient the derivatives of the mass and stiffness matrices are not required.

<table>
<thead>
<tr>
<th>Parameter vertex solution theorem</th>
<th>Central value</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 \times 10^5 )</td>
<td>3.651</td>
<td>3.355</td>
<td>4.020</td>
</tr>
<tr>
<td>( \lambda_2 \times 10^6 )</td>
<td>7.336</td>
<td>6.696</td>
<td>8.030</td>
</tr>
<tr>
<td>( \lambda_3 \times 10^7 )</td>
<td>4.822</td>
<td>4.396</td>
<td>5.271</td>
</tr>
<tr>
<td>( \lambda_4 \times 10^8 )</td>
<td>2.581</td>
<td>2.343</td>
<td>2.811</td>
</tr>
<tr>
<td>( \lambda_5 \times 10^8 )</td>
<td>8.920</td>
<td>8.094</td>
<td>9.714</td>
</tr>
<tr>
<td>( \lambda_6 \times 10^9 )</td>
<td>2.740</td>
<td>2.484</td>
<td>2.980</td>
</tr>
</tbody>
</table>

Table 3: Comparison of eigenvalue intervals for the stepped bar example, case 2

4 The fuzzy eigenvalue problem

4.1 Standard procedure

The general procedure of fuzzy finite element vibration analysis is presented as follows:

I. Establish the fuzzy eigenvalue problem;

II. Set \( m \) thresholds \( \alpha_k, \ k = 1,2,\ldots,m \). For each \( \alpha_k \), transform the fuzzy eigenvalue problem into interval equations using the \( \alpha \)-cut;
III. Calculate the eigenvalue intervals for the $\alpha_k$-cut using the interval analysis method;

IV. Change the value of $k$, repeat steps II and III, and obtain the eigenvalue intervals for each $\alpha_k$, $k = 1, 2, \ldots, m$;

V. Transform the interval eigenvalues obtained in step IV into the fuzzy eigenvalue representation for the original fuzzy eigenvalue problem.

4.2 Interval increment method

Here we present a new interval increment method for fuzzy eigenvalues as follows:

I. Establish the fuzzy eigenvalue problem;

II. Set $m$ thresholds $\alpha_k$, $k = 1, 2, \ldots, m$. For each $\alpha_k$, transform the fuzzy eigenvalue problem into interval equations using the $\alpha$-cut;

III. The parameter interval $b'_{\alpha} = [\underline{b}, \overline{b}]_{\alpha}$ for the $\alpha$-cut $\alpha_k$ is divided into three intervals, $[\underline{b}_{\alpha_{k-1}}, \overline{b}_{\alpha_{k-1}}]$, $[\underline{b}_{\alpha_k}, \overline{b}_{\alpha_k}]$ and $[\underline{b}_{\alpha_{k+1}}, \overline{b}_{\alpha_{k+1}}]$. The interval eigenvalue solution for $[\underline{b}_{\alpha_{k-1}}, \overline{b}_{\alpha_{k-1}}]$ is obtained by the interval solution in the previous step with $\alpha$-cut $\alpha_{k-1}$. In this step, two interval eigenvalues with interval parameters $[\underline{b}_{\alpha_k}, \overline{b}_{\alpha_k}]$ and $[\underline{b}_{\alpha_{k+1}}, \overline{b}_{\alpha_{k+1}}]$ are calculated via a perturbation method. Then the eigenvalue interval is the union of the three intervals.

IV. Change the value of $k$, repeat steps II and III, and obtain the eigenvalue intervals for $\alpha_k$, $k = 1, 2, \ldots, m$;

V. Transform the interval eigenvalues obtained in step IV into the fuzzy eigenvalue representation for the original fuzzy eigenvalue problem.

5 Numerical example

Consider a cantilever beam which is 800mm long, 50mm wide and 25mm deep, shown in Figure 2. The material parameters are $\rho = 7800$ kg/m$^3$ and $E = 210$ GN/m$^2$ and the beam is modelled using 8 beam elements. A pinned support position is uncertain, as shown in Fig. 2, and the beam is analysed using the method described by Friswell [12]. Figure 3 shows the variation in the first two natural frequencies as the support position is varied. The membership function is $[0.55/0.6/0.7]$, as shown in Fig. 4. Figure 5 and 6 show the membership functions of lowest two eigenv alues, estimated both by the standard procedure and by the interval increment method for fuzzy eigenvalues, compared with the exact solution given by Friswell [12]. Using the standard fuzzy solution procedure the accuracy of the eigenvalue interval is poor for the lower $\alpha$-cut values. The solution by the interval increment method is very accurate.

![Figure 2: Cantilever beam example with a pinned support](image-url)
Figure 3: Eigenvalues for the cantilever beam example

Figure 4: The membership function for the uncertain support position
6 Conclusion

This paper has investigated strategies to efficiently solve the fuzzy eigenvalue problem. A new perturbation method has been proposed to produce tighter bounds on the eigenvalues. This method can give the exact supremum and the infimum of the first-order deviation of the structural eigenvalue, which is more efficient than the standard perturbation method. This method is more general than many of the interval arithmetic methods as it does not require the non-negative decomposition of the mass and stiffness matrix pair. The methods to solve the interval analysis problem were summarized and the paper then investigated strategies to efficiently calculate the fuzzy eigenvalue problem. The paper presented a new increment method, in which the parameter intervals to reuse interval results from previous $\alpha$-cuts. The method was demonstrated on a simple cantilever beam with a pinned support, and produced very accurate fuzzy eigenvalues.
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References