Torsional vibration of machines with gear errors

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ABSTRACT

Vibration and noise induced by errors and faults in gear meshes are key concerns for the performance of many rotating machines and the prediction of developing faults. Of particular concern are displacement errors in the gear mesh and for rigid gears these may be modelled to give a linear set of differential equations with forced excitation. Other faults, such as backlash or friction, may also arise and give non-linear models with rich dynamics. This paper considers the particular case of gear errors modelled as a Fourier series based on the tooth meshing frequency, leading immediately to non-linear equations of motion, even without the presence of other non-linear phenomena. By considering the perturbed response this system may be modelled as a parametrically excited system. This paper motivates the analysis, derives the equations of motion for the case of a single gear mesh, and provides example power spectra of a boiler feed pump.

1 INTRODUCTION

Gear systems are very common, and practically impossible to replace in various applications where mechanical power must be transferred. The dynamic behaviour of systems involving gears can be very complex, and phenomena of interest include time varying mesh stiffness due to multiple teeth contact, a backlash between the teeth, and excitation due to displacement errors in the gears [Ozguven, 1988, Theodossiades, 2000, Velex, 1996, Liu, 2008, Litak, 2005]. In consequence, under a dynamic load, a typical gear system is a nonlinear oscillator, exhibiting a range of complex behaviour [Lees, 1978]. As reported by Lees and Haines [1978] this includes fluctuating torque signals at a fixed frequency, unrelated to shaft speed. Such phenomena can only be explained by non-linearities. In the literature random noise is often included, but in the present paper we seek a more rigorous approach.

In practice it is important to minimise the effect of gear tooth errors, to ensure machine stability and to minimise noise and gear wear. But like all physical systems, some error is inevitable and it is therefore important to understand the way in which a system will behave with a real gearbox. During operation wear will develop and at some stage the mating teeth may lose contact during some parts of the operational cycle: this will give rise to further significant non-linearities, but in the paper we discuss the simpler situation in which the gears remain in contact.
When a torsional model of a complete shaft train is being prepared, it is often necessary to include a model of a gearbox. The effect of non-unity gear ratios on torsional vibration is examined. We focus on a relatively simple system, with two shafts linked through a single gear although the derivation of the equations of motion is easily extended to the general case.

### 2.1 Applying constraints for geared systems

Consider the torsional behaviour in a system involving a gear drive, depicted schematically in Figure 1. The system is modelled using four rotary inertias and two torsional springs. The $i$th gear inertia is $I_i$, the working radius of gear $i$ is $R_i$, the $i$th shaft stiffness is $k_i$ and the rotation of the $i$th gear is $\phi_i$. The angular displacements, $\phi_2$ and $\phi_3$, of the inertias representing the gears are coupled through the gear mesh, and are therefore not independent. Let $R_2$ and $N_2$ be the working radius and the number of teeth respectively for gear wheel 2. Similarly, let $R_3$ and $N_3$ be the working radius and the number of teeth respectively for gear wheel 3. If the gear pair is perfect, then the tangential velocities of the two gears are identical. For the present purposes, it is necessary only to recognise that

$$\frac{R_2}{N_2} = \frac{R_3}{N_3}.$$ 

Equating the tangential velocities yields $R_3\dot{\phi}_3 = -R_2\dot{\phi}_2$, where $\dot{\phi}_2$ and $\dot{\phi}_3$ represent the instantaneous angular velocities of gear wheels 2 and 3 respectively. Defining $\gamma$ to be the gear ratio $\gamma = \frac{N_2}{N_3} = \frac{R_2}{R_3}$, then $\dot{\phi}_3 = -\gamma \dot{\phi}_2$. The negative sign indicates that a positive $\dot{\phi}_2$, causes negative $\dot{\phi}_3$. With appropriate choice of initial conditions, we can deduce that $\dot{\phi}_3 = -\gamma \dot{\phi}_2$.

The four equations of motion of this simple system, assuming viscous damping in the shaft, are $\ddot{\phi}_2$.

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**Figure 1. Four-inertia idealization of a drive system**

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\[ I_1 \ddot{\phi}_1 + c_1 (\dot{\phi}_1 - \dot{\phi}_2) + k_1 (\phi_1 - \phi_2) = T_1 \]
\[ I_2 \ddot{\phi}_2 + c_1 (\dot{\phi}_2 - \dot{\phi}_1) + k_1 (\phi_2 - \phi_1) = T_2 + R_{23}F_{23} \]
\[ I_3 \ddot{\phi}_3 + c_2 (\dot{\phi}_3 - \phi_4) + k_2 (\phi_3 - \phi_4) = T_3 + R_{3}F_{23} \]
\[ I_4 \ddot{\phi}_4 + c_2 (\dot{\phi}_4 - \phi_3) + k_2 (\phi_4 - \phi_3) = T_4 \] (1)

where \( F_{23} \) represents the instantaneous force acting between gears 2 and 3 along the common tangent and where \( \{T_1, T_2, T_3, T_4\} \) are the instantaneous torques from external sources being exerted on inertias 1, 2, 3 and 4 respectively. It appears anomalous at first that \( F_{23} \) contributes positively to both gear wheels but inspection of the free body diagrams for disks 2 and 3, shown in Figure 2, reveals that this is correct.

**Figure 2. The free body diagrams for disks 2 and 3 of the four-to include a model of a gearbox inertia idealisation of a drive system**

There are four rotation co-ordinates in Equation (1) but only three of these are independent because of the constraint between \( \phi_2 \) and \( \phi_3 \). We choose \( \{\phi_1, \phi_2, \phi_4\} \) as independent coordinates, and \( \phi_3 \) is eliminated using \( \phi_3 = -\gamma \phi_2 \). The forcing term \( F_{23} \) in Equation (1) is unknown, and this is removed by subtracting \( \gamma \) times the third equation from the second. The resulting three equations in \( \{\phi_1, \phi_2, \phi_4\} \) are

\[ I_1 \ddot{\phi}_1 + c_1 (\dot{\phi}_1 - \dot{\phi}_2) + k_1 (\phi_1 - \phi_2) = T_1 \]
\[ (I_2 + \gamma^2 I_3) \ddot{\phi}_2 + c_1 (\dot{\phi}_2 - \phi_1) + c_2 (\gamma^2 \phi_2 + \gamma \phi_4) + k_1 (\phi_2 - \phi_1) + k_2 (\gamma^2 \phi_2 + \gamma \phi_4) = T_2 - \gamma T_3 \] (2)
\[ I_4 \ddot{\phi}_4 + c_2 (\dot{\phi}_4 + \gamma \phi_2) + k_2 (\phi_4 + \gamma \phi_2) = T_4 \]

These equations are equivalent to those obtained from a three-inertia model of a simple rotor but comprising only three inertias. Suitable mass, damping and stiffness matrices can be extracted from Equations (2) and these may be used to find natural frequencies and mode shapes. A similar approach may be applied to any gear arrangement consisting of multiple gear ratios.

Note that the angles \( \phi_i \) used thus far are defined as the angular positions of the gears and inertias and hence are a combination of the steady state rotation and the torsional vibration. To separate the two angles we define
\[
\phi_i = \Omega_i t + q_i
\]  
(3)

Where \(\Omega_i\) is the steady state rotation of inertia \(I_i\) and \(q_i\) is the torsional vibration. Thus \(\Omega_1 = \Omega_2, \; \Omega_3 = \Omega_4\) and \(\Omega_3 = -\gamma \Omega_2\).

Thus, Equation (2) becomes, in matrix form,

\[
M \ddot{q} + C \dot{q} + K q = \begin{cases} 
T_1 \\
T_2 - \gamma T_3 \\
T_4 
\end{cases}
\]  
(4)

where

\[
q = \begin{bmatrix} q_1 \\
q_2 \\
q_3 \\
q_4 
\end{bmatrix}, \quad M = \begin{bmatrix} I_1 & 0 & 0 \\
0 & I_2 + \gamma^2 I_3 & 0 \\
0 & 0 & I_4 
\end{bmatrix}, \quad C = \begin{bmatrix} c_1 & -c_1 & 0 \\
-c_1 & c_1 + \gamma^2 c_2 & \gamma c_2 \\
0 & \gamma c_2 & c_2 
\end{bmatrix},
\]

\[
K = \begin{bmatrix} k_1 & -k_1 & 0 \\
-k_1 & k_1 + \gamma^2 k_2 & \gamma k_2 \\
0 & \gamma k_2 & k_2 
\end{bmatrix}.
\]

2.2 Displacement-driven excitation of torsional vibration

Thus far, each individual inertia initially has one independent rotation coordinate and at each gear the constraint is of the form \(\dot{\phi}_3 = -\gamma \dot{\phi}_2\). With torsional models in particular, it is very often necessary to model displacement-driven excitation. The most important example of this is gear geometry errors but other cases arise in misaligned coupled rotors. Consider the two meshing gears given in Figure 1. If no gear geometry errors are present, then the ratio \(\phi_2 / \phi_3\) is constant and, with suitable definition of angles such that \(20 = 0\) when \(30 = 0\), it follows that \(N_2 \phi_2 + N_3 \phi_3 = 0\). In the presence of gear geometry errors, the ratio \(\phi_2 / \phi_3\) will not be constant and the constraint equation is modified to

\[
N_2 \phi_2 + N_3 \phi_3 = \varepsilon(\phi_2, \phi_3)
\]  
(5)

where \(\varepsilon\) is the mesh error. Most papers consider this error to be represented as a function of time or frequency depending on the analysis being carried out. However the mesh errors will in practice be non-linear functions of the gear angles, as discussed now.

The factor \(N_2 \phi_2 / 2\pi\) represents the number of teeth on gear \(I_2\) passing through the mesh as the gear rotates though an angle \(\phi_2\). Suppose that the teeth on gear \(I_2\) are not equally spaced but have errors. Then this factor is modified to give \((N_2 \phi_2 - e_2(\phi_2)) / (2\pi)\) where \(e_2\) is a periodic function of \(\phi_2\). The same analysis may be performed for gear \(I_3\). Hence Equation (5) becomes, assuming no interaction between the mesh errors on the two gears,

\[
N_2 \phi_2 + N_3 \phi_3 = e_2(\phi_2) + e_3(\phi_3)
\]  
(6)
or dividing by \( N_3 \), we have

\[
\gamma \phi_2 + \phi_3 = \frac{1}{N_3} \{ e_2(\phi_2) + e_3(\phi_3) \}
\]  

(7)

If the functional form of the displacement errors is known then Equation (7) is a constraint equation linking the angles at the gear mesh, \( \phi_2 \) and \( \phi_3 \). If we could find a closed form expression for \( \phi_3 \) in terms of \( \phi_2 \), or vice-versa, then this constraint may be used directly in Equation (1) to obtain a three degree of freedom model in a similar approach to Section 2.1. However in general this is not possible and an alternative approach must be developed. Here we consider an approximate constraint approach for small vibrations.

### 2.3 Defining gear errors

To proceed the gear errors \( e_2(\phi_2) \) and \( e_3(\phi_3) \) have to be specified. One convenient option is to write the gear errors as Fourier series in \( \phi_2 \) and \( \phi_3 \). In particular the terms relating to the fundamental frequency and the harmonic term corresponding to the tooth meshing are likely to be dominant. Thus, \( e_2 \) is periodic in \( \phi_2 \) and hence,

\[
e_2 = \sum_{n=1}^{\infty} \alpha_{2n} \cos(n\phi_2 + \beta_{2n}) = \sum_{n=1}^{\infty} \alpha_{2n} \cos(n\Omega_2 t + nq_2 + \beta_{2n})
\]  

(8)

for constants \( \alpha_{2n} \) and \( \beta_{2n} \). Similarly \( e_3 \) is periodic in \( \phi_3 \) and hence

\[
e_3 = \sum_{n=1}^{\infty} \alpha_{3n} \cos(n\phi_3 + \beta_{3n}) = \sum_{n=1}^{\infty} \alpha_{3n} \cos(n\Omega_3 t + nq_3 + \beta_{3n})
\]  

(9)

for constants \( \alpha_{3n} \) and \( \beta_{3n} \).

Of course defining the errors in this way means that it is impossible to solve for either \( \phi_2 \) or \( \phi_3 \) in closed form and therefore the constraints cannot be applied explicitly. Often the non-linearity is ignored in displacement driven excitation of geared systems and in this case, for example for \( e_2 \),

\[
e_2 = \sum_{n=1}^{\infty} \alpha_{2n} \cos(n\Omega_2 t + \beta_{2n}).
\]  

(10)

The gear errors are now solely a function of time, and this merely leads to forcing terms in the linear equations of motion.

### 2.4 Assuming Small Vibrations

Equations (8) and (9) express the gear errors as non-linear functions of the gear angles. One approach is to assume that the vibration responses, \( q_i \), are small and ignore second and higher order terms. Thus the series for \( e_2 \) may be expanded using trigonometric identities to give
\[ e_2 = \sum_{n=1}^{\infty} \alpha_{2n} \left\{ \cos(n\Omega_2 t + \beta_{2n}) - nq_2 \sin(n\Omega_2 t + \beta_{2n}) \right\} \]  
(11)

which results in forcing and parametric terms at the frequency \( \Omega_2 \) and its harmonics. Similarly for \( e_3 \)

\[ e_3 = \sum_{n=1}^{\infty} \alpha_{3n} \left\{ \cos(n\Omega_3 t + \beta_{3n}) - nq_3 \sin(n\Omega_3 t + \beta_{3n}) \right\}. \]  
(12)

The constraints in terms of \( \phi_2 \) and \( \phi_3 \) can be written in terms of \( q_2 \) and \( q_3 \) using

\[ N_2\phi_2 + N_3\phi_3 = (N_2\Omega_2 + N_3\Omega_3)t + N_2q_2 + N_3q_3 = N_2q_2 + N_3q_3 \]  
(13)

From Equation (6), substituting Equations (11) and (12), and collecting together terms in \( q_2 \) and \( q_3 \) we have

\[ [N_2 + D_2(t)]q_2 + [N_3 + D_3(t)]q_3 = E_2(t) + E_3(t) \]  
(14)

where

\[ D_2(t) = \sum_{n=1}^{\infty} \alpha_{2n}n \sin(n\Omega_2 t + \beta_{2n}) \quad E_2(t) = \sum_{n=1}^{\infty} \alpha_{2n} \cos(n\Omega_2 t + \beta_{2n}) \]

\[ D_3(t) = \sum_{n=1}^{\infty} \alpha_{3n}n \sin(n\Omega_3 t + \beta_{3n}) \quad E_3(t) = \sum_{n=1}^{\infty} \alpha_{3n} \cos(n\Omega_3 t + \beta_{3n}) \]

Clearly these equations define the time dependent gear ratio as

\[ \gamma(t) = \frac{N_2 + D_2(t)}{N_3 + D_3(t)} \]  
(15)

which results in a parametrically excited system. There is also an excitation term

\[ e(t) = \frac{E_2(t) + E_3(t)}{N_3 + D_3(t)}. \]  
(16)

Thus the constraint equation is

\[ q_3 = -\gamma(t)q_2 + e(t). \]  
(17)

Hence the equation of motion, with no external excitation, is

\[
\begin{bmatrix}
0 \\
I_2 \ddot{\gamma} + c_2 \dot{\gamma} + k_2 \gamma
\end{bmatrix}
+ \begin{bmatrix}
C(t) + \dot{\gamma}(t)C_1(t) \\
K(t) + \dot{\gamma}(t)K_1(t) + \ddot{\gamma}(t)K_2(t)
\end{bmatrix}[q] = \left\{ \begin{array}{c}
I_2 \ddot{\gamma} + c_2 \dot{\gamma} + k_2 \gamma \\
c_2 \dot{\gamma} + k_2 \gamma
\end{array} \right\}
\]  
(18)
where the mass, damping and stiffness matrices, $M$, $C$ and $K$, are defined in Equation (4), with the understanding that $\gamma$ and hence the matrices, are now time dependent. The additional damping and stiffness matrix terms are given by

$$
C_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\gamma(t)I_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma(t)c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma(t)I_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Note that the model has a rigid body mode, which can make a time simulation difficult since the absolute angles of the whole machine can drift together. One option is to impose motion at some point in the machines, for example $q_1(t) = 0$, which changes the boundary conditions of the machine and hence the natural frequencies. A second option is to add a grounded spring of low stiffness to one of the inertias, which will also change the machine dynamics slightly. The alternative, adopted here, is to simulate the full system of equations and look at relative angles, for example $q_2 - q_1$.

### 3 SIMULATED EXAMPLE

The parameters for the simulated system are

$$I_1 = 7000\text{kgm}^2, \quad I_2 = 4000\text{kgm}^2, \quad I_3 = 13000\text{kgm}^2, \quad I_4 = 5000\text{kgm}^2,$$

$$k_1 = 1.6\text{GNm/rad}, \quad k_2 = 0.29\text{GNm/rad}, \quad c_1 = 36\text{MNms/rad}, \quad c_2 = 5.8\text{MNms/rad},$$

$$N_2 = 56, \quad N_3 = 111.$$

The natural frequencies of the linear system are 0, 39.74 and 107.1 Hz.

The gear errors are assumed to be

$$\alpha_{2i} = \frac{0.01}{i}\text{rad} \quad \text{for } 1 < i < N_2, \quad \beta_{2i} = 0 \quad \text{for all } i$$

$$\alpha_{3i} = \frac{0.01}{i}\text{rad} \quad \text{for } 1 < i < N_3, \quad \beta_{3i} = 0 \quad \text{for all } i$$

Higher terms in the Fourier series are assumed to be zero. The largest error is at the rotation speed of the corresponding shaft, for example due to errors in the location of the gear centre. The next highest component is at the harmonic corresponding to the number of teeth and represents tooth meshing errors. The other harmonic terms are significantly smaller.

From the Fourier series of the gear errors it is clear that the excitation contains many frequencies. Furthermore the parametric excitation terms means that there will also be responses at many combination frequencies. Suppose that the spin speed of the first shaft is $\Omega_2 = 90\text{Hz}$. Figure 3 shows the low frequency part of the FFT of the time response of the twist in the first shaft, and clearly shows many combination frequencies. There are large discrete responses at $\Omega_2$ and...
\( \Omega_3 = -45.4 \, \text{Hz} \). For comparison Figure 4 shows the zoomed FFT plot when the parametric excitation is switched off. Clearly the combination frequencies do not now appear. Because the system has reached a steady state there is no excitation and little response at the resonance frequencies of the system.

**Figure 3.** Low frequency response of the twist in the first shaft for the full model \((q_2 - q_1)\)

**Figure 4.** Low frequency response of the twist in the first shaft for the full model \((q_2 - q_1)\) without parametric excitation

Figure 5 shows a waterfall plot as the spin speed of the first shaft is varied. The response at \( \Omega_2 \) and \( \Omega_3 \), and also the sidebands corresponding to the combination frequencies, are clearly visible. There is also a low frequency response corresponding to a combination frequency.
Figure 5. Waterfall plot of the low frequency twist in the first shaft for the full model as the spin speed of the first shaft ($\Omega_1$) varies

4 CONCLUSIONS

This paper has derived a model to simulate the torsional response of a machine due to gear errors. For errors on both gears in a gear mesh the constraint implicit in the displacement driven excitation cannot be implemented exactly. Approximate models were proposed, assuming small vibrations, that lead to parametric excitation. The complicated nature of this parametric excitation mean that a large range of frequencies are excited, and this was demonstrated in a simulated example. The broadband nature of the excitation and response mean that diagnosis of faults in geared systems is very difficult. Fortunately, the vibration amplitudes of the higher frequency response background are relatively small compared to the main low frequency peaks, which could lead to the legitimate derivation of further approximate models. The presence of combination response frequencies in the lower frequency range was also demonstrated. Further investigation of the dynamics of this model are planned to understand properties of gear systems with a view to deriving improved condition monitoring techniques.

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REFERENCES


Friswell MI, Penny JET, Garvey SD and Lees AW 2010, *Dynamics of Rotating Machines*, Cambridge University Press.