Optimum Design of a PID Controller for the Adaptive Torsion Wing Using GA

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This paper presents the optimum design of a PID controller for the Adaptive Torsion Wing (ATW) using the genetic algorithm (GA) optimiser. The ATW is a thin-wall, two-spar wingbox whose torsional stiffness can be adjusted by translating the spar webs in the chordwise direction inward and towards each. The reduction in torsional stiffness allows external aerodynamic loads to deform the wing and maintain its shape. The ATW is integrated within the wing of a representative UAV to replace conventional ailerons and provide roll control. The ATW is modelled as a two-dimensional equivalent aerofoil using bending and torsion shape functions to express the equations of motion in terms of the twist angle and plunge displacement at the wingtip. The full equations of motion for the ATW equivalent aerofoil were derived using Lagrangian mechanics. The aerodynamic lift and moment acting on the aerofoil were modelled using Theodorsen’s unsteady aerodynamic theory. The equations of motion are then linearized around an equilibrium position and the GA is employed to design a PID controller for the linearized system to minimise the actuation power required. Finally, the sizing and selection of a suitable actuator is performed.

Nomenclature

\( \hat{a} \) = normalized pitch axis location with respect to half chord
(\( \hat{a} = -1 \) leading edge, \( \hat{a} = 1 \) trailing edge)

\( A \) = Enclosed area

\( c \) = Chord

\( ds \) = Infinitesimal segment along the perimeter

\( e \) = Distance between aerodynamic centre and shear centre

\( E \) = Young’s modulus

\( G \) = Shear modulus

\( h \) = Average depth of the wing

\( I_y \) = Second moment of area of the wing

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The Adaptive Torsion Wing (ATW) is a thin-wall closed-section wingbox (Fig. 1) whose torsional stiffness can be adjusted through the relative chordwise position of the front and rear spar webs [1-2]. Instead of applying a torque directly to the wing structure to twist it and maintain its deformed shape, the ATW uses the airflow energy to twist the wing and maintain its shape which might be a more energy efficient way of deforming the structure. Nevertheless, some form of actuation is still required to move the spar webs in the chordwise direction and alter the torsional stiffness. Furthermore, the use of the ATW to achieve aeroelastic twist can have significant impact on the flutter and divergence margins as the torsional stiffness and the position of the elastic axis varies with the position of the webs. These aeroelastic deformations can be used in a beneficial manner to enhance the control authority or the flight performance of the air-vehicle. The torsional stiffness ($K_o$) of the ATW, assuming linear twist variation along span of cantilever wing, can be estimated using the 2nd Bredt-Batho equation as

$$ K_o = \frac{GJ}{l} = \frac{4GA^2}{lS \frac{ds}{t_{eq}}} $$

where $G$ is the shear modulus, $J$ is the torsion constant, $l$ is the wing semi-span, $A$ is the enclosed area, $t_{eq}$ is the equivalent wall thickness, and $ds$ is an infinitesimal segment along the perimeter. If a single material is used in the wingbox, the resulting torsional stiffness only depends on the square of the enclosed area; therefore the torsional stiffness of the section can be altered by varying the chordwise position of either: the front spar web, the rear spar web, or both, to change the enclosed area, as shown in Figure 1. The change in web positions results in two components of torsional stiffness; the first component is due to the closed section while the second comes from the skin/web segments belonging to the open section(s). The analysis in this paper accounts for the closed-section component, because the torsional stiffness associated with the open section segments is several orders of magnitude smaller.
The Adaptive Torsion Wing (ATW) is integrated in the UAV wing as shown in Fig. 1b to replace conventional ailerons and provide roll control. The use of the ATW results in lower Radar Cross-Section (RCS) of the vehicle as no discrete surfaces are required. To achieve a roll manoeuvre, the web positions on one side of the wing are rearranged to allow the airflow to twist it and maintain the deformed profile, while the web positions on the other side of the wing are unaltered. This results in a differential lift that generates a rolling moment. The rolling moment or rate desired can be controlled by adjusting the web positions. The representative UAV considered is a high endurance medium altitude vehicle similar to the BAE Systems Herti UAV as shown in Figure 2. The UAV has a rectangular unswept, untwisted, uniform wing. The front web is originally located at 20% of the chord and the rear web is fixed at 70% of the chord. The specifications of the UAV are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wing area</td>
<td>22.44 m²</td>
</tr>
<tr>
<td>MTOW</td>
<td>800 kg</td>
</tr>
<tr>
<td>Cruise speed</td>
<td>60 m/s</td>
</tr>
<tr>
<td>Design dive speed</td>
<td>≈ 82 m/s</td>
</tr>
<tr>
<td>Span (2l)</td>
<td>12 m</td>
</tr>
<tr>
<td>Chord (c)</td>
<td>1.87 m</td>
</tr>
<tr>
<td>Wing loading</td>
<td>35.70 kg/m²</td>
</tr>
<tr>
<td>Aerofoil</td>
<td>NACA 4415</td>
</tr>
</tbody>
</table>

### II. Dynamical modelling

Previous studies showed that the variation in torsional stiffness and elastic axis position associated with shifting the spar webs can have a significant impact on the divergence and flutter boundaries of the wing \([1,2]\). Furthermore, the actuation power depends on the relative position of the webs and the dynamic pressure \([2]\). The wing, shown in Fig. 2, is modelled as an equivalent two-dimensional aerofoil as shown in Fig. 3 using bending and torsional modal shape functions. The bending shape, \(f(y)\), corresponds to a uniform cantilever beam with uniform distributed load along its length, while the torsion shape function, \(\phi(y)\), corresponds to a uniform cantilever beam with linear twist along its length. These shape functions are

\[
f(y) = \frac{y^2(6l^2 + y^2 - 4ly)}{3l^4}
\]

and
where $y$ is the spanwise position measured from the wing root. Using these shape functions, the ATW can be modelled as an equivalent two-dimensional aerofoil whose position is defined by the plunge displacement and twist at the tip of the wing. The plunge and pitch displacements at any spanwise location can be related to the tip displacements by

$$w = w_t f(y) \quad (4)$$

and

$$\theta = \theta_t \phi(y) \quad (5)$$

where $w$ is the plunge displacement at any spanwise position, $w_t$ is the plunge displacement at the wingtip, $\theta$ is the twist angle at any spanwise position, and $\theta_t$ is the twist angle at the wingtip. The equivalent aerofoil is modelled as a uniform bar with the webs represented as point masses that can slide along the bar to change the torsional stiffness and shear centre position. Thus when the autopilot of the UAV commands a rolling moment or rolling rate, this is converted into translations of the webs to provide the aeroelastic twist necessary to generate the rolling moment or rolling rate demanded. The full equations of motion of the equivalent two-dimensional model (Table 1) are developed using Lagrangian mechanics.

A. Conventions and Assumptions

Various conventions and assumptions are adopted throughout the derivation process. These are listed as:

- Four degrees of freedom are required to govern the system. These degrees of freedom are: position of front web ($x_1$), position of the rear web ($x_2$), tip plunge displacement ($w_t$), and the tip twist angle ($\theta_t$);
- The origin is considered to be located at the quarter chord of the equivalent aerofoil. The shear centre was avoided as the origin because its position depends on the relative positions of the spar webs which will increase the complexity of the analysis;
- The forces on the webs are positive when pointing outboard (away from the origin);
- $x_2$ is positive when the rear web is behind the aerodynamic centre;
- $x_1$ is positive when the front web is in front of the aerodynamic centre;
- Small twist angle ($\theta_t$) is assumed hence, $\cos(\theta_t) \approx 1$ and $\sin(\theta_t) \approx \theta_t$;
- The aerofoil is symmetric, hence the aerodynamic pitching moment at the aerodynamic centre $M_a$ is zero during the steady state phase;
- The front and rear webs are represented as point masses.
- Frictional losses are neglected;
- The gravitational potential energy is neglected;
- Webs and aerofoil are rigid bodies, i.e. elastic deformations are neglected;
- The shear centre position always lies half way between the front and rear webs. This results in a maximum 5% error in the shear centre position;
- The aerofoil is a uniform bar where the thickness of this bar is equal to the sum of the equivalent upper and lower skins. The equivalent skins include the covers, spar caps, and stringers. The thickness of the equivalent skin is equal to the thickness of the material required to take the bending and torsional loads;
- The torsional stiffness depends only on the square of the area enclosed between the skins/covers and the webs;
- Thin-wall beam theory is valid.
The parameters of the ATW investigated here are listed in Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of elasticity (E)</td>
<td>72 GPa</td>
</tr>
<tr>
<td>Polar moment of inertia (G)</td>
<td>27 GPa</td>
</tr>
<tr>
<td>Web thickness (t_w)</td>
<td>0.4 mm</td>
</tr>
<tr>
<td>Skin thickness (t_s)</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Mass of front web (m_1)</td>
<td>0.17 kg/m</td>
</tr>
<tr>
<td>Mass of front web (m_2)</td>
<td>0.17 kg/m</td>
</tr>
<tr>
<td>Mass of aerofoil (m_αf)</td>
<td>5.20 kg/m</td>
</tr>
<tr>
<td>Effective depth (h)</td>
<td>0.15 m</td>
</tr>
</tbody>
</table>

A schematic of the ATW equivalent aerofoil is given in Fig. 3. The equivalent aerofoil is modelled as a uniform bar with the webs represented as point masses that can slide along the bar to change the torsional stiffness and shear centre position. Thus when the autopilot of the UAV commands a rolling moment or rolling rate, this is converted into translations of the webs to provide the aeroelastic twist necessary to generate the rolling moment or rolling rate demanded. Lagrange’s equations of motion for a system with multiple degrees of freedom are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial \dot{q}_i} = Q_i
\]  

(6)

where \( T \) and \( U \) are the total kinetic and total potential energies of the system, \( q_i \) represents the \( i^{th} \) degree of freedom, \( \dot{q}_i \) is the first time derivative of the \( i^{th} \) degree of freedom, and \( Q_i \) is the \( i^{th} \) applied force or moment.

### B. Kinetic Energy

The kinetic energy per unit span of the ATW concept is the sum of the individual kinetic energies per unit span of the front web (\( T'_1 \)), rear web (\( T'_2 \)), and the aerofoil (\( T'_{\alpha f} \))

\[
T' = T'_1 + T'_2 + T'_{\alpha f}
\]  

(7)

The kinetic energy term due to the rotation of the webs is neglected for both the front and rear webs.

i) **Front web**

The position of the front web with respect to the origin is

\[
\overline{r}_1 = -x_1 \hat{i} + (w_1 f(y) + x_1 \theta_1 f(y)) \hat{j}
\]

(8)
where $x_1$ is the distance between the origin and the front web and $\vec{i}$ and $\vec{j}$ are directions defined in Fig. 3.

Then the velocity of the front web is

\[ \vec{V}_1 = -\dot{x}_1 \vec{i} + (\ddot{w}_t f(y) + \dot{x}_1 \dot{\theta}_t \phi(y) + x_1 \dot{\theta}_t \phi(y)) \vec{j} \] (9)

Hence the kinetic energy per unit span of the front web is

\[ T'_1 = \frac{1}{2} m_1 \| \vec{V}_1 \|^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 (\ddot{w}_t f(y))^2 + \frac{1}{2} m_1 \dot{x}_1^2 (\dot{\theta}_t \phi(y))^2 + m_1 x_1 \dot{w}_t f(y) \dot{\theta}_t \phi(y) \]
\[ + \frac{1}{2} m_1 \dot{x}_1^2 (\dot{\theta}_t \phi(y))^2 + m_1 \dot{x}_1 \dot{w}_t f(y) \dot{\theta}_t \phi(y) + m_1 x_1 \dot{x}_1 \dot{\theta}_t \dot{\phi}(y) \] (10)

where $m_1$ is the mass per unit span of the front web.

ii) Rear web

The position of the rear web with respect to the origin is

\[ \vec{r}_2 = x_2 \vec{i} + (\dot{w}_t f(y) - x_2 \dot{\theta}_t \phi(y)) \vec{j} \] (11)

where $x_2$ is the distance between the origin and the rear web.

The velocity of the rear web is

\[ \vec{V}_2 = \dot{x}_2 \vec{i} + (\ddot{w}_t f(y) - \dot{x}_2 \dot{\theta}_t \phi(y) - x_2 \dot{\theta}_t \phi(y)) \vec{j} \] (12)

Hence the kinetic energy per unit span of the rear web is

\[ T'_2 = \frac{1}{2} m_2 \| \vec{V}_2 \|^2 = \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 (\ddot{w}_t f(y))^2 + \frac{1}{2} m_2 \dot{x}_2^2 (\dot{\theta}_t \phi(y))^2 - m_2 x_2 \dot{w}_t f(y) \dot{\theta}_t \phi(y) + \]
\[ \frac{1}{2} m_2 \dot{x}_2^2 (\dot{\theta}_t \phi(y))^2 - m_2 \dot{x}_2 \dot{w}_t f(y) \dot{\theta}_t \phi(y) + m_2 x_2 \dot{x}_2 \dot{\theta}_t \dot{\phi}(y) \] (13)

where $m_2$ is the mass per unit span of the rear web.

iii) Aerofoil

The kinetic energy per unit span of the aerofoil is

\[ T'_{af} = \frac{1}{2} m_{af} (\ddot{w}_t f(y))^2 + \frac{1}{2} I_{ac} (\dot{\theta}_t \phi(y))^2 - \frac{1}{4} m_{af} c \dot{w}_t f(y) \dot{\theta}_t \phi(y) \] (14)

where $m_{af}$ is the mass of the equivalent aerofoil per unit span, $I_{ac}$ is the mass moment of inertia of the aerofoil about the aerodynamic centre (origin).

The total kinetic energy of the ATW becomes

\[ T = \int_a^b T' \, dy \] (15)

C. Elastic potential energy

The elastic potential energy consists of two main components: translational and rotational. The rotational potential energy of the system is

\[ U_\theta = \frac{1}{2} K_\theta \dot{\theta}_t^2 \] (16)

where $K_\theta$ is the torsional stiffness of the ATW and using the 2nd Bredt-Batho equation for a uniform cantilever wing can be expressed as
where \( h \) is the average depth of the wing.

The above expression is only valid for a thin-wall rectangular closed section where the equivalent thickness of the aerofoil is equal to the thicknesses of the front and rear webs. The translation potential energy of the system is

\[
U_w = \frac{1}{2} K_w (w_t - e \theta_t)^2
\]

where \( e \) is the distance separating the shear centre and the aerodynamic centre given by

\[
e = \frac{x_2 - x_1}{2}
\]

and the bending stiffness is

\[
K_w = E l_y \int_0^l \left( \frac{d^2 f}{dy^2} \right)^2 dy = \frac{16 E l_y}{5 l^3}
\]

where \( E \) is the Young’s modulus and \( l_{xx} \) is the second moment of area of the wing (equivalent aerofoil). Then, the total elastic potential energy of the system becomes

\[
U = U_{\theta} + U_w = \frac{1}{2} K_{\theta} \dot{\theta}_t^2 + \frac{1}{2} K_w (w_t - e \theta_t)^2
\]

D. Equations of motion (EOMs)

After determining the expressions for the kinetic and potential energies of the system, the four Lagrange’s equations with respect to each degree of freedom can be obtained. The integral of the shape functions are computed and substituted into the equations of motion. The first equation, relative to the position of the front web \((x_1)\), is

\[
m_1 \dddot{x}_1 \left(l + \frac{\theta_t^2 l}{3} \right) + \frac{13}{45} m_1 \dddot{\theta}_t l + \frac{2}{3} m_1 x_1 \dddot{\theta}_t + \frac{1}{2} m_1 \dddot{x}_t \theta_t + \frac{1}{2} \frac{\partial K_{\theta}}{\partial x_1} \theta_t^2 - \frac{1}{2} K_w \dddot{w}_t \theta_t - \frac{1}{2} K_w e \dddot{\theta}_t^2 = F_1
\]

where \( F_1 \) is the force acting on the front spar web. The second equation, relative to the position of the rear web \((x_2)\), is

\[
m_2 \dddot{x}_2 \left(l + \frac{\theta_t^2 l}{3} \right) - \frac{13}{45} m_2 \dddot{\theta}_t l + \frac{2}{3} m_2 x_2 \dddot{\theta}_t + \frac{1}{2} m_2 \dddot{x}_t \theta_t + \frac{1}{2} \frac{\partial K_{\theta}}{\partial x_2} \theta_t^2 - \frac{1}{2} K_w \dddot{w}_t \theta_t - \frac{1}{2} K_w e \dddot{\theta}_t^2 = F_2
\]

where \( F_2 \) is the force acting on the rear spar web. The third equation, relative to the tip plunge displacement \((w_t)\), is

\[
\text{\scriptsize{\frac{104}{405} (m_1 + m_2 + m_{af}) \dddot{w}_t l + \frac{13}{45} (2m_1 \dddot{x}_1 - 2m_2 \dddot{x}_2) \dddot{\theta}_t l + \frac{13}{45} (m_1 x_1 - m_2 x_2 - \frac{1}{4} m_{af} c) \dddot{\theta}_t l +}}
\]

\[
\frac{13}{45} (m_1 \dddot{x}_1 - m_2 \dddot{x}_2) \dddot{\theta}_t l + K_w \dddot{w}_t - K_w e \dddot{\theta}_t = L
\]

where \( L \) is the lift force acting at the quarter chord of the aerofoil. The fourth equation, relative to the tip twist angle \((\theta_t)\), is

\[
\text{\scriptsize{\frac{1}{3} (m_1 x_1 \dddot{x}_1 + m_2 x_2 \dddot{x}_2) \dddot{\theta}_t l + \frac{13}{45} (m_1 x_1 + m_2 x_2 + \frac{2 m_{af} c}{48}) \dddot{\theta}_t l + \frac{13}{45} (m_1 x_1 - m_2 x_2 - \frac{1}{4} m_{af} c) \dddot{w}_t l +}}
\]

\[
\frac{1}{3} (2m_1 \dddot{x}_1 + 2m_2 \dddot{x}_2) \dddot{\theta}_t l - K_w \dddot{w}_t + (K_0 + K_w e^2) \dddot{\theta}_t = M_o
\]

where \( M_o \) is the pitching moment at the quarter chord of the aerofoil.
III. Unsteady aerodynamics

Theodorsen’s unsteady aerodynamic theory was employed for the aerodynamic predictions. Theodorsen’s unsteady aerodynamics consists of two components: circulatory and non-circulatory [3]. The non-circulatory component accounts for the acceleration of the fluid surrounding the aerofoil while the circulatory component accounts for the effect of the wake on the aerofoil and contains the main damping and stiffness terms. The work of Theodorsen is based on the following assumptions:

- Thin aerofoil;
- Potential, incompressible flow;
- The flow remains attached, i.e. the amplitude of oscillations is small;
- The wake behind the aerofoil is flat.

The total Theodorsen’s lift and pitching moment about the shear centre for the two-dimensional model are

\[
L = \pi \rho \frac{c^2}{4} \left[ -\bar{\omega}_t \int f^2 \, dy + \bar{e} \theta_t \int f \phi \, dy + \frac{(V + 2\bar{e})}{2} \theta_t \int f \phi \, dy + \left( e - \bar{\alpha} \frac{c}{2} \right) \bar{\theta}_t \int f \phi \, dy \right] + 2\pi \rho V \frac{c}{2} C(k) \left[ -\bar{\omega}_t \int f^2 \, dy + \left( \bar{\alpha} \bar{e} - V \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int f \phi \, dy \right] + \left( e + \bar{\alpha} \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy + \left( e + \bar{\alpha} \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy + \left( e + \bar{\alpha} \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy
\]

\[
M = \pi \rho \frac{c^2}{4} \left[ -\bar{\alpha} \frac{c}{2} \bar{\omega}_t \int f \phi \, dy + \bar{\alpha} \frac{c}{2} \bar{e} \theta_t \int f^2 \, dy + \left( \bar{\alpha} \bar{e} - V \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int f \phi \, dy + \left( e - \bar{\alpha} \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy \right] + 2\pi \rho V \frac{c}{2} \left( \frac{1}{2} + \bar{\alpha} \right) C(k) \left[ -\bar{\omega}_t \int f \phi \, dy + \left( \bar{\alpha} \bar{e} - V \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy + \left( e + \bar{\alpha} \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t \int \phi^2 \, dy \right]
\]

where \( \rho \) is the air density, \( c \) is the chord of the two-dimensional model, \( \bar{\alpha} = \frac{2e}{c} - \frac{1}{2} \) is the normalized pitch axis location with respect to half chord, and \( C(k) \) is the Theodorsen’s transfer function that accounts for attenuation of lift amplitude and phase lag in lift response due to sinusoidal motion. It should be noted that \( M_o \) is independent of Theodorsen’s transfer function \( C(k) \). The aerodynamic moment acting about the aerodynamic centre becomes

\[
M_o = M - Le
\]

or

\[
M_o = \pi \rho \frac{c^2}{4} \left[ -\frac{2}{5} \left( \frac{\bar{\alpha} \frac{c}{2} - e}{2} \right) \bar{\omega}_t l + \frac{1}{2} \left( \bar{\alpha} \frac{c}{2} e - e \bar{e} \right) \theta_t l + \frac{1}{2} \left( \bar{\alpha} \frac{c}{2} e - V \frac{c}{2} \left( \frac{1}{2} - \bar{\alpha} \right) \right) \bar{\theta}_t l + 2e \bar{e} \theta_t l + \frac{1}{2} \left( 3\bar{e}^2 - e^2 - \frac{e^2}{c^2} \bar{\alpha} \frac{c}{2} + \bar{\alpha} \frac{c}{2} \right) \bar{\theta}_t l \right]
\]

This paper uses the low-dimensional state-space representation of the classical unsteady aerodynamic model of Theodorsen developed by Brunton and Rowley [4]. They employed a Padé approximation of Theodorsen’s transfer function which was used to develop reduced order model for the effect of synthetic jet actuators on the forces and moments on an aerofoil [4,5]. The transfer function \( C(s) \) is approximated by

\[
C(s) \approx \frac{0.5177a^2s^2 + 0.2752as + 0.01576}{a^2s^2 + 0.3414as + 0.01582}
\]

where

\[
a = \frac{c}{2V}
\]

The lift force then becomes

\[
L = \frac{0.0075069}{a^2} u + \frac{0.99845}{a} \dot{u} + \frac{1}{2} (A_L \ddot{e} + 0.5176B_L \ddot{e} + 0.5176B_L V \theta_t l + \frac{1}{2} (A_L (V + 2\dot{e}) + \cdots
\]

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In the lift force equation, Theodorsen’s transfer function \(C(s)\) has 2 poles, and hence two state variables are required to model the transfer function in state space. \(\ddot{u}\) can be obtained from the following differential equation:

\[
\ddot{u} = -\frac{0.3414}{a} u - \frac{0.01582}{a^2} u + \frac{1}{2} B_L (V + \dot{e}) \theta_t l + \frac{1}{2} B_L \left( e + \frac{c}{4} \left( 1 - 2 \bar{a} \right) \right) \theta_t l - \frac{2}{5} B_L \ddot{\theta} l
\]

Equation (35) is treated as an equation of motion with \(u\) as an apparent degree of freedom.

**IV. Linearization of the ATW**

Previous studies on the ATW showed that the rear web is less effective than the front web in producing large aeroelastic twist required for flight control purposes [1,2]. Therefore in this analysis the rear web is fixed in its original position (70% of the chord), and hence Equation (3) and all of \(\dot{x}_2\) and \(\ddot{x}_2\) terms in the EOMs can be eliminated. The EOMs are nonlinear and therefore designing a PID controller for the system is a complex process as the system can’t be represented by a single transfer function whose poles define the stability margins. Furthermore, the nonlinear system might suffer from hysteresis which is beyond the scope of this work. When the nonlinear equations of motion are arranged in state space format, it points out that the nonlinear problem is stiff and using the conventional numerical integrators available in MATLAB™ is computationally inefficient.

To simplify the analysis and have a good approximation of the controller’s coefficients, the EOM are linearized around an arbitrary equilibrium position. The linearization is performed used Taylor’s Series expansion of functions. After linearization, the equation of motion relative to front web position becomes:

\[
m_1 \ddot{x}_1 \left( l + \theta_t^2 \frac{l}{3} \right) + \frac{13}{45} m_1 \dot{x}_{1o} \ddot{\theta}_t l + m_1 x_{1o} \dot{\theta}_t \ddot{x}_1 l \frac{l}{3} + \frac{1}{2} K_w \theta_{t_o} \Delta w_t + K_s \theta_{t_o} \Delta \theta_t + \frac{1}{2} K_w \theta_{t_o} \Delta \theta_t = \Delta F_1
\]

and the equation of motion relative to the tip plunge displacement becomes:

\[
\frac{104}{405} (m_1 + m_2 + m_a) \ddot{w}_t l + \frac{13}{45} (m_1 x_{1o} - m_2 x_{2o} - \frac{1}{4} m_a c) \ddot{x}_1 l + \frac{13}{45} m_1 \ddot{x}_{1o} l + K_w \Delta w_t
\]

\[
- K_w e_o \Delta \theta_t + \frac{K_w \theta_{t_o} ^2 \Delta x_1}{4} = \Delta L
\]

where

\[
\Delta L = \frac{0.0075699}{a^2} \Delta u + \frac{0.09845}{a} \ddot{u} (0.5176 B_L V) \Delta \theta_t^l l + \left( A_L e_o + 0.5176 B_L \left( e + \frac{c}{2} \left( 1 - \bar{a} \right) \right) \right) \frac{l}{2} + \left( A_L e_o - A_L \frac{c}{2} \bar{a} \right) \frac{\ddot{\theta} l}{2} - \frac{2}{5} A_L \ddot{w}_t l
\]
\[ \ddot{u} = -0.3414 \frac{a}{\Delta u} \dot{u} - 0.01582 \frac{a^2}{\Delta u} B_L V \Delta \theta_t \frac{l}{2} - \frac{B_L \theta_{to} \dot{x}_l}{4} + B_L \left( e_o + \frac{c}{4} (1 - 2 \dot{\Delta}_{o}) \right) \dot{\theta}_t \frac{l}{2} - \frac{2}{5} B_L \dot{\omega}_t l \]  

(39)

The equation of motion relative to the tip twist angle becomes

\[ \frac{13}{45} m_1 x_{1o} \theta_{to} \dot{x}_1 l + \frac{1}{3} (m_1 x_{1o}^2 + m_2 x_{2o}^2 + l_{oc}) \dot{\theta}_l l + \frac{13}{45} (m_1 x_{1o} - m_2 x_{2o} - \frac{1}{4} m_{af} \omega_l l

- K_{w} e_o \Delta \omega_t + K_s (x_{1o} + x_{2o}) \Delta \theta_t + K_{w} e_o \theta_{to} \Delta x_1 + K_{w} e_o^2 \Delta \theta_t

+ \frac{1}{2} K_{w} w_{o} \Delta x_1 = \Delta M_o \]  

(40)

where

\[ \Delta M_o = A_L \left[ -\frac{2}{5} \left( \dot{a}_0 \frac{c}{2} - e_o \right) \dot{\omega}_t l + \left( -V \frac{c}{2} \frac{1}{2} - \dot{a}_o \right) - V e_o \right] \dot{\theta}_t \frac{l}{2} + \left( 3 \dot{a}_o \frac{c}{2} e_o - e_o^2 - \frac{c}{4} \right) \right] \]  

(41)

The delta (Δ) notation is used to denote a small disturbance from the corresponding equilibrium position and the (o) subscript is used to denote the values of the variables at equilibrium. The explicit nonlinear expression of the torsional stiffness is simplified for the linear case. Originally the torsional stiffness (K_\theta) using the 2nd Bredt-Batho equation is expressed as:

\[ K_\theta = \frac{2G t_w t_h l^2 (x_1 + x_2)^2}{l \left[ h t_f + (x_1 + x_2) t_w \right]} \]  

(42)

where \( t_f \) is the thickness of the equivalent skin, \( t_w \) is the thickness of the web. In order to simplify the expression the wing’s depth (h) is assumed to be much smaller than the wingbox width (x_1 + x_2) and the skin and web thicknesses are replaced with an equivalent wall thickness (t_eq). The torsional stiffness expression becomes:

\[ K_\theta = \frac{2G t_{eq} h^2 (x_1 + x_2)}{l} = K_s (x_1 + x_2) \]  

(43)

where

\[ \frac{t_{eq} h + \frac{1}{2} c t_w}{h + \frac{1}{2} c} \]  

(44)

The term \( \frac{1}{2} c \) corresponds to the width of the wingbox when the spar webs are located at their original positions (20% and 70% of the chord). After simplification, the relation between torsional stiffness and front web position (x_1) becomes linear.

V. **PID controller for the linearized system**

PID controllers have been widely used in many industrial and aerospace control systems for several decades. The first paper on PID controller appeared in 1922 by the Russian-American engineer Nicolas Minorsky while developing automatic steering of ships [6]. This is because the PID controller structure is simple and its principle is easier to understand than most other advanced controllers. In industrial applications, more than 90% of all control loops are PID type [7], and PID control can achieve satisfactory performance in most cases. To ensure an optimum performance and accurate response of the ATW system, a PID controller is added to form a closed loop control system as shown in Fig. 4.
The linearized EOMs (before adding a PID controller) and Equation (39) indicates that 8 state variables are required to model the ATW, resulting in a transfer function with 6 zeros and 8 poles. The poles and zeros of the linearized ATW once the front web is released from its original position at 20% of the chord are given in Table 3. It should be noted that the poles and zeros vary depending on the equilibrium position selected for linearizing the EOMs.

Table 3 The poles and zeros of the Adaptive Torsion Wing.

<table>
<thead>
<tr>
<th>Poles</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8.5 + 163i</td>
<td>-8.5 - 163i</td>
</tr>
<tr>
<td>-14.3 + 25.6i</td>
<td>-8.7 - 25.6i</td>
</tr>
<tr>
<td>-11.4 + 3i</td>
<td>-11.4 - 3i</td>
</tr>
<tr>
<td>11.5</td>
<td>-3.4</td>
</tr>
</tbody>
</table>

In Table 3, one of the real poles (11.5) is positive thus the above transfer function is unstable. In fact, a positive real pole indicates that the system is prone to static divergence. The reason for this instability is that at equilibrium, the front web is assumed to be locked in position using a special mechanism and the wing structure is stable, but once the locking mechanism is released, the front web is free to move and the system becomes unstable. By tracking the source of the positive real pole, the linearized stiffness matrix is observed to have one of its eigenvalues negative, which indicates that the system is unstable as its equivalent stiffness drops significantly as the web is released. The existence of instability in the system increases the complexity of designing a PID controller because the controller has to stabilise the system and at the same time provide the targeted system’s response. Therefore designing a controller that can provide optimal system response and ensure stability requires a global search method. A Genetic Algorithm (GA) optimiser and search method was employed due to its ability to deal with such complexity and find a global optimum. The GA incorporated in this paper is based on the “Matlab GA Toolbox”, developed by Chipperfield et al. [8]. A fitness value is assigned to every individual of the initial population through an objective function that assesses the performance of the individual in the problem domain. Power was selected as an objective function because it is a representative figure of merit and depending on the scale or size of the vehicle; power can become the main designer driver, as it can be directly related to weight. Furthermore for small scale vehicles, the power available for actuation can quite limited. Then, individuals are selected based on their fitness index and crossover between them is performed to generate new offspring. Finally, mutation of the new offspring is performed to ensure that the probability of searching any subspace of the problem is never zero. These abovementioned processes iterate until the optimum solution is achieved depending on the convergence criteria of the problem. The PID controller delivers the actuation force on the front web to move the web from one equilibrium position to another while maintaining overall stability and meeting the response requirements of the system. The force on the front web $\Delta F_i(t)$ is the output of the controller and the input for the ATW system while the tip twist angle $\theta_i(t)$ is the system output. In other words, when the autopilot commands a change in the rolling moment or tip twist, the controller provides an actuation force on the web to move it to a new position to meet the demanded tip twist. The closed loop feedback ensures that the achieved tip twist is very close to the desired one. After adding the PID controller to the linearized ATW system, the equations of motion of the ATW are rearranged in state space form as

$$\{q\} = [A]\{q\} + [B]\Delta F_i(t) \tag{45}$$

where $\{q\}$, the state vector, is
\[
\{q\} = \begin{bmatrix}
\Delta w \\
\Delta \theta \\
\Delta u \\
\Delta x_1 \\
\dot{w} \\
\dot{\theta} \\
\dot{u} \\
\dot{x}_1 \\
\end{bmatrix}
\]  \tag{46}

and

\[
\Delta F_i(t) = K_p \ er(t) + K_i \int er(t) + K_d \ \dot{er}(t) \tag{47}
\]

In order to model the PID controller in state space form, a new state variable \(v\) was introduced to account for the integrator term \(v = \int er(t)\) which will add a pole to the closed loop transfer function of the system.

A. A parametric study

This parametric study aims to determine the sensitivity of the actuation power required to the speed of response or the actuation time. In other words, if the ATW is used for roll control of an agile UAV, then the response of the system must be fast and must settle to the targeted tip twist in the shortest period of time possible with minimum overshoot. On the other hand, if the ATW is used for roll control of a high endurance UAV similar to the Herti, then the speed of response and overshoot are of minor concerns. Consider a flying scenario where the Herti is rolling slowly with a tip twist of 0.025 radians. Suddenly, the autopilot commands a targeted tip twist angle of 0.035 radians to perform the manoeuvre faster. This means that a 0.01 radians increase in tip twist must be provided by the controller. The equilibrium state at which the EOMs are linearized is detailed in Table 4.

<table>
<thead>
<tr>
<th>Table 4 The equilibrium position of the wing.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>Tip twist ((\theta))</td>
</tr>
<tr>
<td>Tip plunge ((w))</td>
</tr>
<tr>
<td>Web position ((x_1))</td>
</tr>
<tr>
<td>Angle of attack ((a))</td>
</tr>
<tr>
<td>Airspeed ((V))</td>
</tr>
<tr>
<td>Air density ((\rho))</td>
</tr>
</tbody>
</table>

As stated above, the selection of the PID coefficients is performed using the GA optimiser to minimise the actuation power required to drive the web from the equilibrium position to the new position while meeting other design/response constraints. The design of the PID controller using the GA is performed for different actuation times ranging between 2.5 s to 25 s. Table 5 summarises the optimisation problem.

<table>
<thead>
<tr>
<th>Table 5. Optimisation problem.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
</tr>
<tr>
<td>Variables</td>
</tr>
<tr>
<td>(K_p)</td>
</tr>
<tr>
<td>(K_i)</td>
</tr>
<tr>
<td>(K_d)</td>
</tr>
<tr>
<td>Constraints</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

For each actuation time, the GA runs with 200 generations and 2000 individuals per generation. A large size population was selected, because many of the candidate PID controllers are incapable of stabilising the system and therefore they are given a zero fitness value and disregarded. The variation of actuation power and the controller coefficients with actuation time is shown in Fig. 5. Note that all the coefficients of the controller are negative, but in the figure below, their absolute values were plotted.
Figure 5. The variations of actuation power and PID coefficients for different actuation times.

By examining Fig. 5, the actuation power is highly dependent on the actuation time. For 2.5s the actuation power is about 180kW. In contrast, for 25s actuation time, the actuation power is 1.8kW. This means that increasing the actuation time 10 folds reduces the actuation power 100 fold. The proportional and integrator coefficients drop as the actuation time increases. In contrast, the derivative coefficient is independent of the actuation time or speed of response. When the lower bound of the derivative coefficient is reduced from $10^7$ to $10^6$, this resulted in the system becomes unstable. This explains that the value of $Kd$ remains almost constant for different actuation times to ensure stability of the system.

VI. Actuator sizing and selection

As stated above, the weight and size of the actuation system can have a significant impact on the conceptual design of the vehicle. The power required to drive the webs can be used to estimate the weight and the size of the actuation system given the power density and density of the actuator. The main focus in this section is to estimate the weight and size of the actuator. In order to estimate the weight of the actuator, the worst case actuation scenario shall be considered at the ultimate flight condition. The ultimate flight condition is when the vehicle is cruising at its maximum speed of 60m/s at 3050m and the autopilot commands the largest rolling moment possible which corresponds to a maximum tip twist of 0.1015rad. Initially when the front web is located at its original position (20% of the chord), the aeroelastic twist at the tip is 0.0115 rad when the UAV is cruising at the ultimate flight condition. The equilibrium conditions at the web’s initial position are summarised in Table 6.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tip twist ($\theta$)</td>
<td>0.0115 rad</td>
</tr>
<tr>
<td>Tip plunge ($w$)</td>
<td>0.45 m</td>
</tr>
<tr>
<td>Web position ($x_1$)</td>
<td>0.095 m</td>
</tr>
<tr>
<td>Angle of attack ($\alpha$)</td>
<td>0.034 rad</td>
</tr>
<tr>
<td>Airspeed ($V$)</td>
<td>60 m/s</td>
</tr>
<tr>
<td>Air density ($\rho$)</td>
<td>0.905 kg/m$^3$</td>
</tr>
</tbody>
</table>

The linearized model (PID and ATW) and the GA are used to find the optimum controller coefficient and minimum instantaneous actuation power. The analysis is performed at 9 different equilibrium positions starting from 0.0115 radians up to 0.1015 radians with a step size of 0.01 radians. At each equilibrium position the
autopilot commands a change in the tip twist of 0.01 radian. Two actuation times of 4.5s and 9s are considered. These correspond to a step size of 0.5s and 1s respectively at each equilibrium position. The optimisation problem is summarised in Table 7.

<table>
<thead>
<tr>
<th>Objective function</th>
<th>Minimise (Power)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Variables</strong></td>
<td>$K_p$, $K_d$, $K_i$, $\theta_o$</td>
</tr>
<tr>
<td><strong>Constraints</strong></td>
<td>$-10^8 \leq K_p \leq -10^4$</td>
</tr>
<tr>
<td></td>
<td>$-10^8 \leq K_d \leq -10^4$</td>
</tr>
<tr>
<td></td>
<td>$-10^8 \leq K_i \leq -10^4$</td>
</tr>
<tr>
<td></td>
<td>Overshoot $\leq 25%$</td>
</tr>
</tbody>
</table>

The variation of actuation power and the controller coefficients with the different equilibrium positions are shown in Fig 6.

![Figure 6. The variations of actuation power and PID coefficients for different web positions.](image)

The maximum instantaneous actuation power is required to shift the front web rearward from its initial position (20% chord) to the new position to change the tip twist by 0.01 rad. Then the power drops significantly as the webs moves rearward closer to the rear web (fixed). 21MW of power is required to change the tip twist by 0.01 rad in 0.5s and 5.7MW of power required to change the tip twist by 0.01 rad in 1s. It turns out that the actuation power is large and hence a large actuator would be required. The main reason for this large power requirement is that once the front web is released from its original position, the ATW becomes unstable (Table 3) and hence the actuators is required to stabilise the system and maintain its stiffness by providing a very large
force for a short period of time. In addition, it is not optimal to actuate the front web very fast initially or use the same actuation time for all equilibrium positions. From an actuation point of view it is preferable to actuate the front web relatively slowly initially, and then increase the actuation speed. Since the maximum instantaneous power occurs at 0.0115 rad, it is more feasible to actuate the web from its initial position to the new position in 1s then reduce the actuation time to 0.5s. This will increase the total actuation time by 0.5s (5s total actuation time) but can result in a large weight saving. In order to select the most feasible (minimum weight and size) actuator, three types of actuators are considered in this analysis: hydraulic, pneumatic, and shape memory alloys (SMA). The properties of the actuators (listed in Table 8) are taken from Huber et al. [9].

<table>
<thead>
<tr>
<th>Actuator type</th>
<th>Maximum power density (MW/m$^3$)</th>
<th>Density (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydraulic</td>
<td>500</td>
<td>1600</td>
</tr>
<tr>
<td>Pneumatic</td>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>SMA</td>
<td>50</td>
<td>6500</td>
</tr>
</tbody>
</table>

Table 8. The actuators properties.

For each of the scenarios, the size and weight for each class of actuators is estimated. This allows selecting the most suitable actuator class and indicates what actuation scenario is the most feasible.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Actuator type</th>
<th>Volume (m$^3$)</th>
<th>Weight (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0.5s</td>
<td>Hydraulic</td>
<td>0.0420</td>
<td>67.2</td>
</tr>
<tr>
<td></td>
<td>Pneumatic</td>
<td>4.2</td>
<td>840</td>
</tr>
<tr>
<td></td>
<td>SMA</td>
<td>0.42</td>
<td>2730</td>
</tr>
<tr>
<td>t=1s</td>
<td>Hydraulic</td>
<td>0.0114</td>
<td>18.24</td>
</tr>
<tr>
<td></td>
<td>Pneumatic</td>
<td>1.14</td>
<td>228</td>
</tr>
<tr>
<td></td>
<td>SMA</td>
<td>0.114</td>
<td>741</td>
</tr>
</tbody>
</table>

Table 9. Sizing actuators for the different scenarios.

Table 9 indicates that the hydraulic actuator is the most suitable option for the ATW. 1 kg hydraulic actuator can provide 0.3125 MW of power and hence an 18.24kg actuator is required to actuate the ATW on one side of the wing in an overall time of 5s. In contrast pneumatic and SMA actuators result in significant increase in the MTOW of the UAV which is, in some cases, unrealistic and it can eliminate the benefits of the ATW in providing roll control and enhancing stealth characteristics of the vehicle.

**VII. Conclusion**

The optimum design of a PID controller for the Adaptive Torsion Wing (ATW) using a Genetic Algorithm (GA) to minimise the actuation power was performed. The ATW was employed in a UAV wing to replace conventional ailerons and provide roll control. The wing was modelled as an equivalent two-dimensional aerofoil using bending and torsional shape functions. The full equations of motion (EOMs) were developed using Lagrangian mechanics and Theodorsen’s unsteady aerodynamic theory was employed for aerodynamic predictions. A low-dimensional state-space representation was used to model the Theodorsen’s transfer function and to allow time-domain analysis. The EOMs are linearized using Taylor’s series expansion. A parametric study showed that the actuation power is very sensitive to the actuation time. 21 MW is maximum actuation power required for the ATW. The most suitable actuation system is the hydraulic actuator and an 18.24kg hydraulic actuator is required for the ATW.

**VIII. Future work**

One engineering solution that can prevent the instability problem is by adding a spring to stiffen the structure and stabilise it when the web is released from its locked position. The addition of the spring will reduce the force delivered by the PID controller significantly. The analysis performed in this paper will be repeated with the addition of a suitable mechanical spring.
Acknowledgments
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References