Numerical and experimental analysis of a square bistable plate

A.Carrella¹, M.I. Friswell¹, A. Pirrera¹, G.S. Aglietti²
¹ University of Bristol, Faculty of Engineering, Bristol BS8 1TR, UK
e-mail: A.Carrella@bristol.ac.uk
² University of Southampton, School of Engineering Sciences Southampton SO17 1BJ, UK

Abstract
Multistable composite elements are a convenient approach to realise morphing or shape-adaptable structural systems. This property is particularly important, for example, in the aeronautical industry where morphing structures can be exploited for better aircraft performance and operational versatility. In this paper it is proposed to study the dynamics of a bistable square plate with pinned boundary conditions using a simplified single-degree-of-freedom (SDOF) model. Initially the numerical load-deflection characteristic of the centre of a plate pinned at the four corners is obtained from a finite element model (FEM) using ABAQUS. This curve was then adjusted to account for the change in material properties due to exposure to moisture and others ambient variables. The dynamic response of the plate was simulated by solving the equation of motion numerically. A test rig was designed and built. The bistable plate was hinged to two beams which are rigid in the vertical direction but allow for horizontal displacement. As predicted, the measurements show that the response of the plate to an harmonic excitation of the base is periodic for a low amplitude of excitation. For large excitation, a snap-through (passage from one stable state to the other) takes place. Ideally, if the boundary condition were symmetric, the chaotic passage from one stable state to another could be observed (as for a system with a double-well potential). However, in practice ideal boundary conditions cannot be achieved and the load-deflection characteristic is not symmetric. As a result, the frequency and the amplitude at which the snap occurs depends on the initial stable configuration.

1 Introduction
Morphing or shape-adaptable structural systems are increasingly being considered as a solution to the always present need for better aircraft performance. Such systems should simultaneously fulfil the contradictory requirements of flexibility and stiffness. So far the solutions adopted consist of complex assemblies of rigid bodies hinged to the main structure and actuated. To enhance the performance of the aircraft as a system, multistable composites could provide an interesting alternative to traditional designs, thanks to their multiple equilibrium configurations. Unsymmetric laminates exhibit out-of-plane displacements at room temperature even if cured flat. These displacements are caused by residual stress fields induced during the cool-down process between the highest curing temperature (∼ 160°C) and room temperature (∼ 20°C). The thermal stresses are mainly caused by the mismatch of the coefficients of thermal expansion of constituent layers [1–5]. The main feature of a bistable structure is the snap-through mechanism which marks the passage from one stable position to the other. It is clear that an analytical dynamic analysis of the bistable plate would reveal very complicated behaviour. However, the analysis of an accurate Finite Element (FE) model would have a high computational cost. Therefore, in this paper it is proposed to model the dynamics of the
The snap-through mechanism with a SDOF system, as suggested by the work of Mattioni et al. [6] and Arrieta et al. [7]. The authors justify the SDOF assumption by considering the load-deflection characteristic at the centre of the plate. Such a curve (force applied at the centre vs. its displacement) can be obtained readily with a commercially available Finite Element (FE) package (in this case ABAQUS). The curve shown in Fig. 1 (dashed line) was obtained by pinning the four corners of the plate (but allowing in-plane motion) and applying a quasi-static load at the centre. It shows a displacement range in which the stiffness is negative, and where an oscillatory dynamic response is not possible. Carrella and Friswell [8] exploited the region with negative stiffness to increase the frequency range in which a passive vibration isolator is effective. A bistable system is characterised by a double-well potential. One example of such a system is the Duffing equation (with negative linear term) which has been extensively studied [9–11] and is known to exhibit a linear, a nonlinear (softening) or a chaotic dynamic behaviour.

In this paper the dynamic response of a square bistable plate is studied. The aim is to predict the level of excitation that triggers the snap. In the analysis, only the motion at the centre of the plate is considered and this enable one to model the plate as a SDOF system. The restoring force is first obtained numerically with an FE analysis, and then adjusted to take into account the degradation of the material properties with time and moisture. The equation of motion is solved numerically with Matlab® to calculate the dynamic response in order to establish at what level of excitation, in the frequency range if interest, the plate snaps from one stable state to the other. Subsequently, a rig has been designed and built. The measurements confirm the numerical simulations with a good degree of accuracy.

2 Numerical static and dynamic analysis

The dashed line in Fig. 1 shows the force-deflection curve of a square plate (20×20 cm) with 4-ply asymmetric laminate $[0_2-90_2]^T$ obtained by a finite element analysis. As the force is increased, the curve reaches a peak (at about 20 N). A further increase of the applied force causes the plate to snap. This is due to the negative slope of the curve between the peak and the trough, i.e a negative stiffness coefficient. A system with such a restoring force is known also as ‘double-well potential’ system and can present a chaotic dynamic behaviour [14, 16].

In practice, the material properties of the plate degrade progressively with time. This is mainly due to exposition to moisture which affects the tension in the plies, and hence the load-deflection curve (stiffness). Ideally, the load-deflection curve of the plate should be measured. However, since this has not been possible, a rough estimate of the true snapping load was obtained by loading the centre of the plate with weights until it snapped. It was found that the plate passed from one stable state to the other under a weight of about 1.1 kg (10.8 N). The estimated load-deflection curve is plotted with a solid line in Fig. 1 and has been obtained from the numerical FE curve multiplied by a ‘degradation’ factor of 0.54 (given by the ratio of the maximum measured force to the maximum numerical force necessary to cause the snap).

Fig.2 is the schematic representation of an SDOF system excited at the base. If the base undergoes an harmonic motion with amplitude $Y_0$, the equation of motion is

$$m\ddot{z} + c\dot{z} + f(z) = -m\omega_e^2 Y_0 \sin(\omega_e t)$$  \hspace{1cm} (1)

where $z = x - y$ is the relative displacement between the mass and the base, $c$ is the damping coefficient, $f(z)$ is the restoring force, and $\omega_e$ is the excitation frequency. Eqn.(1) can be solved numerically with Matlab®. In particular, the value for the mass is $m = 0.055$ kg (measured mass of the system tested), whilst the restoring force $f(z)$ is obtained from linear interpolation of the FE load-deflection curve. The damping coefficient was changed depending on the excitation level. The dynamic response of the system was simulated for three different excitation levels over a discrete set of excitation frequencies (20-40 Hz, in increments of 2 Hz). The
amplitude cases were: a) two constant amplitudes of base excitation ($Y_0 = 0.5, 0.75$ mm) and b) a constant level of base acceleration ($\omega_c^2 Y_0 = 2.5$ g). The transmissibility, plotted as a solid line in Fig. 5(a) to 5(c), was calculated as ratio between the root-mean-square (RMS) of the steady-state response to the amplitude of the excitation.

In the next section, experimental results are presented and compared with the simulated results.
3 Dynamic analysis: experimental results

A laboratory scale rig was assembled, as shown in Fig. 3, and placed on a vertical shaker. Fig. 3 shows the plate in one of its stable positions, with a curvature in the $v$ axis. The second stable position has significant curvature only in the $u$ axis. However the boundary condition are not symmetric because the hinges are only placed along the $u$ axis. As a result, the force required for the snap, i.e. change of curvature axis, is smaller when the curvature is in the $v$ direction (as shown in the figure).

![Figure 3: Experimental rig. The bistable plate was hinged to two vertical beams which allowed for a horizontal displacement. The response was measured with an accelerometer at the centre of the plate, whilst the input signal was monitored with an accelerometer at the base.](image)

In order to replicate the numerical simulations, the system was excited at discrete frequencies between 20 and 40 Hz (with a step-increment of 2 Hz) with a constant base displacement amplitude of 0.5 mm and 0.75 mm. At each frequency, once the system had achieved a steady-state, ten seconds of data were captured using an NI DAQPad-6020E data acquisition card. The acceleration of the base was measured using an ENDEVCO 2256-100 accelerometer and the acceleration at the plate centre was measured using a PCB type 352C22 accelerometer. Figures 4(a) and 4(b) show a concatenation of the acquired time histories for each excitation frequency for the two amplitudes of the base displacement. The top plot of Fig. 4(a) shows the amplitude of the base excitation (constant at 0.5 mm) whilst the bottom plot is the response; Fig. 4(b) displays the same data for the case of a base amplitude of 0.75 mm. For this level of excitation the snap does not occur and the system behaves in a rather linear manner. This can be better seen in Figs. 5(a) and 5(b) where the ratios between the root-mean-square (RMS) of the base and the plate accelerations, that is the transmissibility, have been plotted for the two cases considered. Both curves have a peak at 24 Hz. In order to induce the snap, the level of excitation was considerably increased so that the base oscillates harmonically with a constant acceleration of 2.5g at 20, 21, 22 and 23 Hz. The RMS transmissibility is shown in Fig. 5(c) (dashed line). A sudden jump to a lower transmissibility amplitude can be observed. The passage from one stable state to the other occurred at a frequency of 21 Hz as opposed to 22 Hz predicted with the numerical simulations.

In order to assess the degree of linearity of the system, the spectral content of the acceleration time history for different level of excitation was calculated for each excitation frequency. In Figs. 6(a) to 6(c) the spectrum at the peak frequencies (21 and 24 Hz), which correspond to an acceleration of 2.5 and 1.75g respectively, is shown. The amplitude of the higher order harmonics is always significantly smaller than the amplitude of the fundamental.
4 Discussion

The transmissibility has been computed as ratio between the RMS of the output to the input signals. Although this is strictly true for sinusoidal waveforms, the analysis of the harmonic content of the output has shown that the amplitude of the higher order harmonics is significantly less than the amplitude of the fundamental, which allows the response of the plate to be approximated by a sine wave. Figures 5(a) to 5(c) show a good agreement between the simulations and the measurements. However, it should be noted that: 1) by changing the excitation level from 0.5 to 0.75 mm, the resonance frequency has not changed. This can be interpreted as the fact that for such a low level of excitation the stiffness is still constant; 2) the peak transmissibility, which in a linear system at the resonance frequency is inversely proportional to the damping ratio, decreases as the amplitude increases even though the peak frequency is unchanged. This could be due to a nonlinear damping coefficient, but the investigation of this phenomenon goes beyond the scope of this paper. In this paper the damping has been assumed to be of viscous type and, in order to match the response amplitude, the damping coefficient is $c = 0.85, 0.75, 0.55$ for the 3 excitation levels (0.5mm, 0.75mm and 3.5g respectively); 3) at resonance, the maximum acceleration is $a_{max} = \omega^2 Y_0 = 1.16g$ for a 0.5 mm amplitude and $1.75g$ for 0.75 mm, where $\omega$ is the frequency of excitation and $Y_0$ the amplitude of the base displacement. As the amplitude of acceleration is increased to 2.5g the peak response (which correspond to the snap of the plate to the other stable position) occurs at a lower frequency. This agrees somewhat with previous analysis in the literature where a system with a double well potential exhibits a softening dynamic response; 4) finally, the numerical simulation predicted the passage from one stable state to the other with an acceleration of 2.5g at a frequency of 22 Hz. This has been measured to be 21 Hz (which, given the frequency resolution used, indicates a maximum error of 7%).

5 Concluding remarks and future work

The dynamic change of stable configuration of a bistable composite plate has been simulated by solving the equation of motion of a single degree of freedom system with a restoring force obtained from the FE analysis of the structure. The simulations were then reproduced in the laboratory using an experimental rig. Although the measurements proved the numerical analysis to be rather accurate (with an error of 4.7% in the prediction of the frequency at which the snap occurs for an excitation level of 2.5g) there are issues that need to be addressed in future work. The boundary conditions under which the load-deflection curve of the
Figure 5: Numerical (solid line) and experimental (dashed line) RMS transmissibility of the system for different amplitudes of the base displacement (a) 0.5mm and (b) 0.75 mm. Plot (c) shows the transmissibility for a base excitation with a fixed acceleration amplitude of 2.5g.

FE analysis were calculated were substantially different from those of the experimental rig, with the lack of symmetry being a major difference. It is proposed to redesign the rig with some appropriate alternative boundary conditions and to obtain the measured load-deflection characteristic.

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Figure 6: Frequency content of the measured signal at resonance for three different level of acceleration: (a) 24 Hz / 1.75g, (b) 23 Hz / 2g, (c) 21 Hz / 2.5g

References


