Regularization for Symmetric and Almost Symmetric Systems in Model Updating

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Abstract
The occurrence of mechanical systems with symmetries in the context of problems with parametric response dependencies can cause a range of problems such as ill-conditioning, singularities, qualitative changes of behaviour. Common methods using parameterized response model representations include the techniques of model updating. The main driver of these algorithms is often the sensitivity matrix corresponding to a given parameterization. Using only the sensitivity matrix in cases of systems with symmetries will result in erratic behaviour of the updating algorithms, numerical problems and can even cause the algorithms to fail to converge to physically acceptable results. The goal of this paper is to investigate the potential of regularization conditions to stabilize the behavior of problems with symmetries. The use of regularization is demonstrated in a numerical case study with singularities and ill-conditioning present in the first order approximation representation.

1 Introduction

This paper presents a study of regularized model updating of structural systems with symmetries. The type of symmetries considered here is induced by the geometric symmetry of the modelled structure with a symmetric initial model parameterization. An improved understanding of the problem can be gained through the consideration of the response space and the internal geometric relationships that exist within this space. The response space considered here is based on the modal characteristics of the structure under consideration. The general theoretical context of this approach for model updating was presented in [1]. This approach can be also related to the methods developed in the field of nonlinear regression [2]. The main task in model updating is the generation of an improved mathematical representation of the existing structure on the basis of the measured structural responses, for example modal characteristics. This is an inverse problem [3], and hence one needs to address the ill-conditioned and ill-posed nature of inverse problems, [4] and [5]. Furthermore, mechanical systems with symmetries may have parametric response dependencies that can cause numerous problems such as ill-conditioning or singularities, qualitative changes in the behaviour, or multiplicities in the spectral representation of the models. These problems are particularly evident in methods utilising first order approximations of the response models based on the sensitivity matrices [1].

The two broad classes of symmetries can be recognised in this area. The first type of symmetry is that caused by a symmetric parameterization of an already symmetric structure, and is the main subject of interest of the current paper. The second type of the symmetry is related to problems with repeated or degenerate roots in the modal representations, causing further problems or the inability to apply first order approximations of the response models. This second type of problem was investigated in [6] and further refined in [7], and is not considered further in this paper. However, the current paper is intended to provide a framework for more profound and general aspects of symmetry related problems represented by these latter problems. More specifically, the goal of this paper is to investigate the potential of
regularization conditions to stabilize the behaviour of the problems caused by a symmetric parameterization.

The paper consists of a general theoretical introduction presented in the Section 2, that describes the concept of the response space. Section 3 introduces regularization as a tool that may be used to overcome problems with overly sensitive inverse problems with respect to missing or unmeasured information and noise. The concepts of the response space and the regularization are both presented in Section 4, specifically for the response space based on a modal representation of the structure. The consequences of a symmetric parameterization on the updating method, utilising the notion of the response space, is analysed in Section 5. The paper is concluded, in Section 6, with a numerical case study demonstrating the concepts introduced and associated problems.

2 General concepts in model updating

The goal of model updating is the model-based rectification of the differences observed between the modelled and the measured sets of responses. One set of responses is obtained from the model and is described by the response vector \( \mathbf{z} \in \mathbb{R}^{N_z} \), where \( N_z \) is the number of selected responses. A set of corresponding measured responses is denoted by \( \mathbf{z}_m \in \mathbb{R}^{N_z} \), where \( \mathbf{z}_m \) is assumed to be close to the ideal response \( \mathbf{z}_{\text{exact}} \). The response vectors \( \mathbf{z} \) are located in the response space \( \mathbb{R}^{N_z} \).

The model is assumed to have unknown or uncertain physically based parameters that characterize the sources of model inaccuracies. These parameters are assembled into the vector of model parameters \( \mathbf{p} \in \mathbb{R}^{N_p} \), where \( N_p \) is the number of all parameters and \( \mathbb{D}_p \) represents feasible parameter domain. The parameter vectors \( \mathbf{p} \in \mathbb{D}_p \) are located in the parameter space \( \mathbb{R}^{N_p} \). The system model establishes a map, \( \mathcal{Z}(\mathbf{p})=\mathbf{z} \), between the parameter and the response spaces, \( \mathcal{Z}:\mathbb{D}_p \rightarrow \mathbb{R}^{N_z} \).

The problem of model updating can be formalized as the task of finding the parameter vector \( \mathbf{p}_m \) such that \( \mathcal{Z}(\mathbf{p}_m)=\mathbf{z}_m \). This notation indicates the inverse nature of the problem, although, in general, it is neither desirable nor feasible to perform this operation directly. The updating problem is reformulated as an optimization task minimizing the distance between the model and measured responses. The distance measure is specified via the \( \mathbf{W} \)-based Euclidean norm, \( \| \mathbf{x} \|_W = \| \mathbf{x}^T \mathbf{W}^{-1} \mathbf{x} \|_2 \), \( \mathbf{W}>0 \). The baseline problem of model updating is defined as follows

\[
\min d_2^2(\mathbf{z},\mathbf{z}_m) \quad \text{for} \quad d_2^2(\mathbf{z},\mathbf{z}_m) = \| \mathcal{Z}(\mathbf{p}) - \mathbf{z}_m \|_W^2, \quad \mathbf{p} \in \mathbb{D}_p.
\] (1)

The solution of problem (1) may proceed using standard approximation methods utilising Taylor series expansions of optimized distance measure \( d_2^2 \). A local quadratic approximation of \( d_2^2 \) can be used to produce the Gauss-Newton (GN) procedure. This approach results in a linear approximation of the distances, \( \mathcal{Z}(\mathbf{p})=\mathbf{z}_k+\mathbf{S}_k \delta \mathbf{p}_k \), where \( \mathbf{z}_k \) is the reference point in the response space, starting at \( \mathbf{z}_0=\mathcal{Z}(\mathbf{p}_0) \) where \( \mathbf{p}_0 \) is the initial parameter vector, \( \mathbf{S}_k=\mathbf{S}(\mathbf{p}_k)=\mathcal{DZ}(\mathbf{p}_k) \in \mathbb{R}^{N_z \times N_p} \) is the Jacobian or sensitivity matrix, and \( \delta \mathbf{p}_k=\mathbf{p}-\mathbf{p}_k \) is the parameter increment.

The sequential minimization of the GN measure \( d_{2\text{GN}}^2=\| \mathbf{S}_k \delta \mathbf{p}_k - \delta \mathbf{z}_k \|_W^2 \), \( \delta \mathbf{z}_k=\mathbf{z}_m - \mathbf{z}_k \), at the current parameter estimate, \( \mathbf{p}_k \), provides the standard GN formula for the parameter update, assuming \( N_z>N_p \), as

\[
(\mathbf{S}_k^T \mathbf{W}_x \mathbf{S}_k) \delta \mathbf{p}_{k,\text{GN}} = -\mathbf{S}_k^T \mathbf{W}_x (\mathbf{z}_k - \mathbf{z}_m).
\] (2)

The previous developments and formula (2) assume the existence and satisfactory conditioning of the matrix \( \mathbf{S}_k \). Often this is not the case and extensions to (1) or (2) must be applied. The sensitivity matrix gives detailed information about the local topology of the problem solution, including an indication of the quality of the parameterization \( \mathbf{p} \).
3 Regularization in model updating

The main problem associated with the optimization defined in (1), or the direct application of inverse operator $Z^{-1}$, is that the problem is, in general, ill-posed. For a well-posed problem, the solution exists, is unique and is stable [4]. Ill-posed problems fail at least one of these criteria. In the case of (1) or (2) problems arise due to the combined effect of random noise on $z_m$ and the nature and quality of the matrix $S$. This paper focuses on the problems related to the sensitivity matrix $S = S(p)$.

In its original form, equation (2) fails to provide a unique solution when $N_p > N_z$, i.e. when the problem is over-parameterised, and when the matrix $S$ is rank deficient. Furthermore, the matrix $S$ appears in (2) as the consequence of the first order approximation $z_k = z_0 + S_k \delta p_k$ of $Z$ and it consists of the partial derivatives $s_{ij} = \partial z_i/\partial p_j$. This procedure assumes the existence and uniqueness of $s_{ij}$.

To address some of these issues the baseline updating process (1) or (2) is extended by its augmentation by additional information based on a priori observations or assumptions about the problem. This process is called regularization and its purpose is the stabilization of the (pseudo-) solution of the inverse problem. Regularization may be implemented as

$$\min \left( d_z^2(z, z_m) + \alpha r_p^2(p) \right)$$

for parameter based regularization, with $r_p(p)$ introducing constraints related to the vector of updating parameters $p$. The function $r_p(p)$ can be used to express distance, smoothness, symmetry or other assumptions through the measure $r_p : D_p \rightarrow \mathbb{R}$. The scalar value $\alpha \in \mathbb{R}^+$ weights the influence of $r_p$ in the overall composite measure $d_z^2 + \alpha r_p^2$. An alternative to formula (3) can be response-based regularization, where the optimized measure is of the form $d_z^2 + \beta r_z^2$, where $r_z : z(p) \rightarrow \mathbb{R}$ provides complementary assumptions about the structure and the relationships between the responses in the space $\mathbb{R}^{N_z}$.

For practical purposes, similar to the measure $d_z$, the measures $r_p$ and $r_z$ must be either linear or linearised. If, for instance, the structure of the solution is assumed in the form $C p = d_p$, its incremental form can be adopted as $C \delta p = d_p - C p_i = d_i$ and the equivalent form to (2) for the sequentially optimized problem (3) is

$$(S_k^T W_z S_k + \alpha C^T W_p C) \delta p_k = -(S_k^T W_z (z_k - z_m) + \alpha C^T W_p (C p_k - d_p))$$

where positive definite matrices $W_z$ and $W_p$ introduce relative weighting in the spaces $\mathbb{R}^{N_z}$ and $D_p \subseteq \mathbb{R}^{N_p}$.

The introduction of the regularizing term in (3) is to increase the stability of the solution with noisy and/or incomplete measured data, $z_m$, and to tackle ill-conditioning problems associated with the matrix $S_k$. The global weighting factor $\alpha$ should be chosen to ensure a balance between the primary response measure and the secondary regularizing term. However, a choice of suitable $\alpha$ or local $\alpha_i$ may be difficult due to symmetries in the problem/model definition.

4 The eigenvalue response space in model updating

In this paper the response vector $z$ is assumed to consist of only natural frequencies, or similar quantities such as eigenvalues, of a parameterized mathematical model represented by a pair of square matrices $\{M(p), K(p)\}$, where both matrices are of order $N_p$ and symmetric. $M$ is the positive definite mass matrix and $K$ is the positive semi-definite stiffness matrix. Both matrices are assumed to retain their properties for any $p \in D_p$. This matrix pair represents a discrete oscillatory system with equivalent
representation \( \{ \Lambda(p), \Phi(p) \} \), where \( \Lambda \) is the diagonal eigenvalue matrix of order \( N_p \) and \( \Phi \in \mathbb{R}^{N \times N_p} \) is the eigenvector matrix. Both representations are connected through standard parameterized eigenvalue problem \( K(p)\Phi(p) = M(p)\Phi(p)\Lambda(p) \). The use of selected eigenvalues assembled into the response vector \( z \), i.e. \( [z] = [\Lambda_{U_i}] \), where \( U_i \) represents the indices of the eigenvalues, establishes the response model \( Z(p) \) employed in model updating.

It is beneficial to consider \( Z(p) \) in the response space \( \mathbb{R}^{N_z} \) in terms of its geometrical representation and interpretation of some of the previously introduced quantities. For this purpose the matrix \( S_k \) is taken as an assembly of column vectors \( s_{k,i} \in \mathbb{R}^{N_z} \), where \( S_k = [s_{k,i}] \) with \( i = 1, 2, \ldots, N_p \). This set of vectors establishes a non-orthogonal basis at the point \( z_k \) on \( Z(D_p) \subseteq \mathbb{R}^{N_z} \). Locally, if a unique \( S_k \) exists, this representation provides a linear approximation of \( Z(D_p) \) around \( z_k \) such that \( z_{lin} = z_k = \sum_{i=1}^{N_p} s_{k,i} (p_i - p_{k,i}) \).

If \( N_p < N_z \), then \( z_{lin} \) belongs to the affine subspace \( A_k \) (a translated generalisation of the linear subspace), embedded in \( \mathbb{R}^{N_z} \) and tangent to the hyper-surface representing \( Z(D_p) \). Any measured response vector \( z_m \in \mathbb{R}^{N_z} \) is orthogonally projected onto \( A_k \). Equation (2), assuming \( W_z = I \), gives a projected measured measurement \( z_{m,k} = z_k + s_k \Phi_k^{-1} (z_m - z_k) \), where \( s_k \Phi_k^{-1} \) indicates the Moore-Penrose generalised inverse of matrix \( S_k \). The use of \( W_z \neq I \) changes the local metric of this operation and the use of the regularized form (4) changes the position from the reference \( z_{m,k} \) to \( z_{m,k}(\alpha) \). This variable projected measured response \( z_{m,k}(\alpha) \) thus establishes the path \( z_{m,k}(\alpha) = I_z(\alpha) \subseteq A_k \). The qualitative nature of \( I_z(\alpha) \) can be used to investigate the regularization condition and for the selection of the local \( \alpha_k \) specified as optimum in some sense, such that \( \alpha_k = \alpha_{opt} \). One possibility is when a sudden change in the direction of \( I_z(\alpha) \) is perceived, the so-called L-curve criterion, [3].

The case of \( N_p \geq N_z \) constitutes a situation where \( A_k \) potentially spans the whole response space \( \mathbb{R}^{N_z} \) thus leading to the situation \( \|z_{m,k} - z_k \| \to 0 \) on convergence. If \( N_p > N_z \) then the computation of \( z_{m,k} \) also requires additional assumptions to obtain a unique solution. This, and previous configurations, are adversely affected when specific (structural) relationships in the set of basis vectors \( \{s_{k,i}\} \), \( i = 1, 2, \ldots, N_p \), is detected, for example through (close) linear dependencies, symmetries and parameter groups. This is particularly relevant in the over-parameterized cases where the linearised response basis of \( A_k \) can be potentially reduced through the regularizing assumptions thus providing meaningful projections leading to \( z_{m,k}(\alpha) \). The subject of the study pursued here is the under-parameterized case with symmetries.

The matrix \( S_k \) can be treated with analytical tools such as the Singular Value Decomposition (SVD). The sensitivity matrix is then decomposed as follows

\[
S_k = U_k \Sigma_k V_k^T \quad \text{where} \quad U_k U_k^T = I_{N_z}, \quad V_k V_k^T = I_{N_p} \tag{5}
\]

and \( \Sigma_k \) is the rectangular matrix of size \( N_z \times N_p \) with nonzero principal diagonal containing ordered singular values \( \sigma_{k,1} \geq \sigma_{k,2} \geq \ldots \geq \sigma_{k,N_p} \geq 0 \), where \( \pi = N_p \) if \( N_p < N_z \) and \( \pi = N_z \) if \( N_p \geq N_z \).

The matrix product \( S_k^T W_z S_k = [s_{k,i}^T W_z s_{k,i}] \) contains the diagonal elements \( \|s_{k,i}\|_W^2 \). If parameter influences of \( p_i \) and \( p_j \) are \( W \)-orthogonal then the off-diagonal elements representing \( W \)-based vector product are \( s_{k,i}^T W_z s_{k,i} = 0 \). The effect of the local approximation based on \( S_k \) can be further analysed by considering the quadratic part of the function \( d_{z,GN}(\delta p_i) \). Assuming \( W_z = I \) then

\[
\delta z_i^T \delta z_i = \delta p_i^T S_k^T S_k \delta p_i = \delta p_i^T (\Sigma_k^T \Sigma_k) V_k^T \delta p_i = \delta r_i^T (\Sigma_k^T \Sigma_k) \delta r_i = \sum_{i=1}^{N_p} \sigma_{k,i}^2 \delta r_i^2 \tag{6}
\]
where the orthogonal transformation \( T = V^T \delta + P \) is introduced to represent the square of the response residual \( \| \delta z_k \|_2^2 \) in terms of principal coordinates \( \delta v_{k,i} \) along principal vectors \( v_{k,i} \), where \( V = [v_{k,i}] \).

This introduces a quadratic representation of the operator \( S_k \) in the parameter space. Alternatively, a representation in the response space can be assumed based on the relationship between the basis vectors \( U = [u_{k,i}] \) and \( V = [v_{k,i}] \) in the form the decomposition (5). This allows the construction of the principal parameter effects in the response space as \( S_k v_{k,i} = \pm \sigma_{i,j} u_{k,i} \). The principal axes of the quadratic representation in the response space centred at \( z_i \) can be written as

\[
S_k v_{k,i} = \pm \sigma_{i,j} u_{k,i} \quad \text{where} \quad i = 1, 2, \ldots, N_p.
\] 

A similar analysis to the one based on the geometric concepts of the response space, quadratic forms and SVD decomposition can be applied to the regularizing condition \( C \delta p_k = d_k \), where \( d_k = d_p - C p_k \). This approach will be pursued further in the next section. These concepts allow further analysis of effectively bi-criterial problems that are solved in the case of the application of regularizing conditions in the forms represented by equation (3).

5 Symmetries in a model updating context

The sensitivity matrix \( S_k \) plays a critical role in a number of numerical algorithms, e.g. equation (2), and it also provides important information about the local properties of the parameterisation, relationship (7). Using only the sensitivity matrix in the cases of systems with symmetries will result in erratic behaviour of the updating algorithms, numerical problems and can even cause the algorithm to fail to converge to physically acceptable results. Moreover, frequent cases of over-parameterisation with \( N_p \ll N \) induce model configurations with similar parameter effects, which are reflected back as effective linear dependencies between the sensitivity vectors \( s_{k,i,j} \).

These are manifestations of either global or local symmetries in the problem arrangement. In the case of global symmetries the combination of symmetrically arranged structure and symmetric parameterisation (across parameter locations, types and values) creates whole subsets of numerically rank deficient operators \( S_k \). In the case of local symmetry a choice of specific parameterisation can create locally, for certain \( p_\omega \), the situation where the groups of the parameters have identical effects, leading to the problems with \( S_k \).

However, more frequent, and potentially more dangerous cases due to their concealed nature, are near symmetric cases. In practical terms these occur in real-life cases of nominally symmetric structures due to associated factors such as production inaccuracies and errors, asymmetric wear, sensor placement, etc. In this case the structure will have ill-conditioned sensitivity matrices with a detrimental impact on the updating algorithms.

5.1 Symmetric parameterization and its implications

Negative effects associated with near or full symmetries in the parameterization definition generate domains in the feasible part of the parameter space \( D_{p,S} \subset D_p \) with either ill-conditioned or rank deficient matrices \( S(p_{r,S}) \), \( p_{r,S} \in D_{p,S} \). The degree of these problems can be indicated by the condition number, which is defined here as \( CN_i = \sigma_{i,\text{max}} / \sigma_{i,\text{min}} \). In the case of linear dependencies in \( S_k \) this ratio increases from the ideal case \( CN = 1 \) towards the case with singularity \( \sigma_{i,\text{min}} = 0 \), where \( CN \) is undefined.

The symmetry induced problems can be addressed with the help of additional regularizing conditions, and this process is described in Section 3. The additional condition is used to enrich the information basis for
the selection of the next model evaluation point $p_{k+1} = p_k + \delta p_k$. In this section two specific conditions are analysed as these are often used in given contexts.

The first condition is given as $C_1 \delta p_k = d_1$, with $C_1 = I$ and $d_1 = 0$. The matrices here have sizes compatible with the problem definition. The second condition is given as $C_2 \delta p_k = d_{2,k}$, with $d_{2,k} = -C_2 p_k$ and $C_2 = \text{bidig}(1,-1) \in \mathbb{R}^{(N-1) \times N}$. The first condition gives preference to solutions in the proximity of the reference point $p_k$. In the second case, assuming a symmetric arrangement of the reference parameters $p_k$ and the right hand side vector $d_{2,k} = 0$, the preferred solutions are located in null($C_2$). The two forms can be used to complement the reference problem (2) into augmented problem (4). The use of the two conditions results into two different one-parametric solution families $\delta p_{1,k}(\alpha)$ and $\delta p_{2,k}(\alpha)$. A single $\alpha_k$ has to be chosen to provide a unique parameter step. The selection of $\alpha_k$ becomes the key problem in the application of the regularizing condition and one alternative is to evaluate behaviour of the trade-off curves combining residual information from both conditions in the form

$$l_{1,2}(\alpha) = \|d_{1,k}(\alpha)\| + \|d_{2,k}(\alpha)\|, \quad \alpha \in \mathbb{R}^+.$$ (8)

The curve $l_{1,2}(\alpha)$ becomes a part of the updating algorithm and it is used in conjunction with equation (4).

The purpose of this curve is to help to identify the candidate regularisation weights in $\mathbb{R}^+$ by investigating structural properties of the curve such as clearly identifiable directional transitions in the curve. To be able to perform this process automatically within the context of an overall model updating algorithm the interaction between the problems characterised by the pairs $\{S_1, \delta z_1\}$ and $\{C, d_1\}$ has to be understood to provide a clear choice of $\alpha_k$.

The transition from the initial projection $z_{m,1} \in A_1$ provided by the relationship (2) for $\alpha = 0$ through the linearised, regularized responses $z_k + S_k \delta p_k(\alpha)$ establishes a one-parametric path $z_{1,c}(\alpha) \in A_1$. The shape of $z_{1,c}(\alpha)$ is determined by the two matrix pairs $\{S_1, \delta z_1\}$ and $\{C, d_1\}$, and these can be further analysed with the help of the concepts presented in Section 4. The generic regularization pair $\{C, d_1\}$ is represented here by its two realisations $\{1,0\}$ and $\{\text{bidig}(1,-1),0\}$. It can be shown that for $\alpha \to \infty$ the condition based on the pair $\{1,0\}$ promotes $\delta p_{1,k}(\alpha)$ such that $\|\delta p_{1,k}\| \to 0$ and $\delta p_{1,k} \to \rho_k \text{grad}(d_{2,k})$, where $\rho_k \in \mathbb{R}$. Alternatively, the condition $\{\text{bidig}(1,-1),0\}$ for $\alpha \to \infty$ promotes $\delta p_{2,k}(\alpha)$ such that $\delta p_{2,k} \to \text{null}(\text{bidig}(1,-1))$. These two directions can be mapped onto the response space as $\rho_k S_k \text{grad}(d_{2,k})$ and $S_k c_2$, where $c_2 \in \text{null}(\text{bidig}(1,-1))$, respectively.

For the current investigations, four response directions of interest are recognised at any given step $k$: a) the principal parameter effect direction $\pm S_k v_{l,k}$, b) the direction of the orthogonal projection of $z_m$ onto $A_k$ based on equation (2), c) the direction $\rho_k S_k \text{grad}(d_{2,k})$ due to regularisation condition 1, and d) the direction $S_k c_e$ due to regularisation condition 2. The presence of actual or near symmetries in the problem setup introduces additional structure into the problem reflected in the shape of the response surface and the quality of local linear approximations based on $S_k$. There may exist a whole subset of the parameter space $D_{p_S} \subset D_p$ where local linear models $S_k$ are directly affected. Moreover, in the analytical setting it is customary to start from $p_0 \in D_{p_S}$, i.e. from fully symmetric configurations, and the expected result $Z^{-1}(z_m)$, as represented by the measured $z_m$, is also located in or close to $D_{p_S}$. This configuration results in a situation where the path followed by the local approximants $S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_{N_S}$, where $N_S$ is the number of the steps required to achieve convergence, is continuously affected by symmetry-associated effects. Further, particularly for converging scenarios with $k \to N_S$, the symmetry induces similar parameter effects and increasing directional correspondence between the four response directions of interest and thus the discerning capability induced by the interaction between $d_z$ and $r_p$ is lost.
leads to the situation where the features for the selection of $\alpha_k$ on the basis of the curve $l_{T_k}(\alpha)$ are not clearly identifiable. The following case study simulates the scenario described, and demonstrates the associated effects. It also presents the analysis of their consequences on the updating algorithm.

6 Case study: discrete 3-dof system with parametric symmetry

This case study is designed to illustrate some of the problems and the consequences of symmetries in the parameterization in nominally symmetric structures and their influence on the quantities used in regularized model updating. The three degree-of-freedom (DOF) structure consisting of simple mass and spring elements in a symmetrical arrangement is shown in Figure 1. This model can be considered as a representation of the frequent case in Finite Element (FE) based modelling where the structure is constrained to the surrounding environment via uncertain, but nominally identical attachment components. The uncertainty associated with these locations causes a lack of correlation between the measured and modelled responses, including the modal representations of both cases. A low-dimensional model is used here to allow investigations embracing the geometric concepts in the response space previously introduced.

![Figure 1: Baseline 3-DOF spring-mass system](image)

The model consists of the three discrete mass elements with the following parameter values: $m = [m_i] = [1, 2, m_3]$ [kg], where $m_i \in \{1.60, 1.05, 1.00\}$ [kg]. The spring elements in the model have parameter values $k = [k_i] = [k_1, 1500, 1500, 3000]$ [N/m]. The active parameters are chosen to be $p = [p_1, p_2]^T = [k_1, k_2]^T$ and the responses are all three natural frequencies $z = [z_1, z_2, z_3]^T = [f_1, f_2, f_3]^T$. This arrangement creates the updating configuration with the response model $Z : D_p \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, where $D_p = [10, 5000] \times [10, 5000]$ [N/m].

The sensitivity matrices are therefore $S = [s_1, s_2] \in \mathbb{R}^{3 \times 2}$ and the sensitivity vectors $s$, are located in the three-dimensional response space on a 2-parameter surface representing the response surface. Therefore these two vectors constitute non-orthogonal basis of a local linear approximation $A$, effectively represented as a tangent plane located at the specific $z = Z(p)$.

For subsequent analysis three different configurations of the model in Figure 1 are assumed: 1) the case with $m_3 = 1.60$ [kg] representing the asymmetric scenario, 2) the case with $m_3 = 1.05$ [kg] representing the near symmetric scenario and 3) the case with $m_3 = 1.00$ [kg] representing the fully symmetric arrangement. Three parameter points will also be considered for each case, representing: a) the initial model realisation, b) the converging model realisation and c) the “measured” model realisation. Parameter values for these cases are summarised in Table 1. The summary of all relevant responses for all mass perturbation cases and all three model realisation points are assembled in Table 2.
Parameter values Model parameters, \( \mathbf{p} = [p_1, p_2]^T = [k_1, k_2]^T \)

<table>
<thead>
<tr>
<th>Scaled ([-)]</th>
<th>Initial</th>
<th>Converging</th>
<th>Measured</th>
</tr>
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<tr>
<th>Absolute ([\text{kN} \text{m}^{-1}])</th>
<th>Initial</th>
<th>Converging</th>
<th>Measured</th>
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<td>[0.1, 0.1]</td>
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<td>[3.0, 3.0]</td>
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Table 1: Summary of parameter values for all considered model cases

| Response values | Perturbed mass, \( m_i \) |
|---|---|---|
| \([f_1, f_2, f_3]\) \([\text{Hz}]\) | 1.60 [kg] | 1.05 [kg] | 1.00 [kg] |
| Initial | [3.61, 5.77, 9.77] | [4.01, 6.29, 9.98] | [4.04, 6.37, 10.01] |
| Converging | [6.78, 9.13, 11.31] | [7.17, 10.42, 11.56] | [7.20, 10.56, 11.65] |
| Measured | [6.84, 9.21, 11.39] | [7.22, 10.54, 11.64] | [7.25, 10.68, 11.72] |

Table 2: Summary of applied responses for all considered response cases

To illustrate some of the geometrical concepts from Section 4, the response space and the response surface plot is shown in Figure 2 for the case with near symmetry, \( m_3 = 1.05 \) [kg]. The space of the three natural frequencies represents the response surface for \( \mathbf{p} \in \mathbb{D}_\mathbf{p} \). The subset \( \mathbb{D}_{p,s} \subset \mathbb{D}_\mathbf{p} \) is, in this case, represented as a part of the line \( k_1 = k_2 \) and corresponding part of the response surface is indicated by the ridge including the points \( z_0, z_k \) and \( z_m \).

![Figure 2: 2-parametric response surface with local linear models given by characteristic ellipses](image)

Since the perturbation of \( m_3 \) is small, the two halves of the surface are so close to each other that they are almost unrecognisable in the figure. The parameter effect lines are shown as the grid of thin solid gray lines. At every intersection of this grid a model linearization is indicated by the ellipse characterising the
The numerical nature of the corresponding sensitivity matrix $S_k$. The cases with near “orthogonal” parameter effects are represented as near-circles, while cases with parametric dependencies are visualised as narrow ellipses, indicating increasing condition number $CN_k$ of the corresponding $S_k$. The information about the condition number for the whole space $D_p$ and all three $m_3$-perturbation cases is summarised in Figure 3. The condition number contour maps in this figure are used to provide clear information about the development of this quantity that indicates numerical problems.

The case with the asymmetric arrangement of the masses clearly indicates the absence of symmetry. However, quasi-symmetric order can be observed, emerging away from the main symmetry diagonal and indicated by increasing condition numbers along a line almost parallel to the line $k_1=k_4$. This effect represents the emergence of parameter dependencies once the spring $k_4$ is stiffened to compensate for the increase in the mass $m_3$. Thus even significantly asymmetric cases can behave locally as the symmetric case, due to subsystem dynamic symmetrisation as indicated by the increase in the condition number towards the observed $CN_{\text{max}}=26.4$.

![Figure 3: Condition number maps for three different mass configurations](image)

The case with the small parametric perturbation with $m_3=1.05$ [kg] is given by central subplot in Figure 3. This case clearly indicates the presence of symmetries and the maximum observed condition number in $D_p$ is $CN_{\text{max}}=157.3$. In this case the contours of the subplot closely match the line of parametric symmetry $k_1=k_4$ (dashed lines) with only a small deviation away from this line.

The final case has complete parametric symmetry with $m_3=1.00$ [kg] giving exact symmetry, and the condition number maps has a maximum value of $CN_{\text{max}}\to\infty$ for the domain $D_{p,\omega}$ ($k_1=k_4$). This represents the cases of a rank deficient $S_\omega$ for $p_\omega\in D_{p,\omega}$, i.e. in this case $\text{rank}(S_\omega)=1$ or $s_1=\gamma s_2$, $\gamma=1$. The wider region around $D_{p,\omega}$, however, is also adversely influenced and characterised by large condition numbers.

The use of a regularizing condition is particularly critical for the third case where equation (2) would fail to provide $\delta \mathbf{p}_4$. In the other two cases the application of equation (2) would provide a unique solution, however, the use of the regularizing condition may still be beneficial to ensure control over the magnitude and the direction of the parameter increments $\delta \mathbf{p}_4$. 
The case based on the use of regularization condition 1, i.e. \( \{1,0\} \), is presented in Figure 4. This figure shows the normalised trade-off curves \( l_{T\delta}(\alpha) \), equation (8), for all three mass perturbation cases. Each subplot contains two lines, where the solid line represents the case based on the linearisation at the point \( z_0 \) and the dashed line represents the case based on the linearisation at the “converging” point \( z_k \). The relative position of these two points with respect to \( z_m \) is provided in Figure 2. Both axes of all subplots of Figure 4 are scaled with respect to the maximum values of the corresponding residuals \( d_{Z,max} \) and \( r_{P,max} \).

The most promising result in terms of the ability to identify a specific change of the direction on \( l_{T\delta}(\alpha) \) and thus identify \( \alpha_k \) is shown for the case \( m_3=1.05 \) [kg] and the linearisation at \( z_0 \). The corresponding lines in the other mass perturbation cases, however, do not indicate the same effect. In the case with \( m_3=1.60 \) [kg] this is due to the relatively well-conditioned nature of \( S_0 \), i.e. a circle-like characteristic ellipse, leading to a progressive transition from \( z_{m,k} \) without any sudden change in direction. In the case with \( m_3=1.00 \) [kg], the absence of a clear change in the direction is due to the singular nature of the characteristic ellipse, which in this case collapses to a line, representing \( \text{rank}(S)=1 \) in the case of the matrix \( S \in \mathbb{R}^{3 \times 2} \).

![Figure 4: Normalised trade-off curves for the three mass configurations and two model realizations (solid: initial model, dashed: converging model) for parameter condition 1](image)

The cases of the dashed lines in Figure 4 indicate consistent behaviour of the trade-off curve on convergence. This arises because the path of the linearised-regularized solution \( z_k + S_k \delta p_k(\alpha) \) does not indicate any directional changes as all relevant parameter effect directions are almost aligned. This indicates the closely coinciding nature of “local” features, e.g. gradients, and “global” features, e.g. the initial projection of \( z_m \) onto \( A_k \), \( z_{m,k} \). More specifically, in the case of \( \{1,0\} \) regularization, this is due to the close alignment of parameter effect directions \( \rho_k S_k \text{grad}(d_k^2) \) and \( S_k^* \delta z_k \).

A similar problematic situation can be observed in the case of \( \{\text{bidiag}(1,-1),0\} \) regularization. However, in this case the path in \( A_k \) is a straight line due to the requirements of equation (4), as represented by the path \( z_k + S_k \delta p_k(\alpha) \) from \( z_{m,k} \) to \( S_k^* c_2 \), \( c_2 \in \text{null(} \text{bidiag(1,-1)} \)\), can be accomplished by a straight line. This line starts at \( z_{m,k} \), and is orthogonally oriented towards \( S_k^* c_2 \), as a consequence of the minimum-distance nature of the overall problem at the \( k \)-th step in the response space. The outcome of this situation is shown in Figure 5 where the original straight paths in the response space are mapped onto only slowly varying \( l_{T\delta}(\alpha) \) curves for both cases of initial \( z_0 \) and converging \( z_k \) based linearisations.
This case study demonstrates the effects associated with the symmetries due to combined symmetric arrangements of both modelled systems and applied parameterisations. These problems result in similar parameter effects in the symmetric configurations giving further ill-conditioning problems or complete loss of directional information within the defined mathematical context as specified by the dimensions of the matrices and the vectors applied in the problem.

![Figure 5: Normalised trade-off curves for two model realizations and parameter condition 2](image)

The negative consequences of these configurations can be partially treated through the application of additional regularizing conditions. These conditions complement the reference problems with free regularizing/weighting parameters. The task at each step of the updating algorithm is to choose suitable values of these parameters, such that the balance between two contributing influences is achieved. The ancillary tool in this process uses the trade-off curves. However, as demonstrated by the case study, these methods give indications of suitable regularizing weights only when strong directional dependence is present in the linearised model, shown by differing strengths of its major and minor principal parameter effects. Thus there must be identifiable blocks in the group of singular values of the sensitivity matrix for a given linearisation point in the response space. Otherwise, these criteria show only non-specific trade-off curves representing slow transition from the reference to the regularizing sub-problems.

7 Conclusion

The paper presents an analysis of model updating of structures with close or exact symmetries in their parametric representations. These problems can be expected to be affected by singularity and ill-conditioning problems when handled by the standard first order methods utilising sensitivity matrices. More specifically, the effect of these symmetric configurations on regularizing terms and the associated regularization parameter choice has been investigated and demonstrated in the case study.

The concept of the response space is applied throughout the studies. The response space has been specified as the domain of primary problem definition accompanied with a simple scalar distance measure representing the match between the two compared models. The process of regularized parameter steps is explained and considered. In general, the regularization has the potential to counteract negative consequences of symmetric arrangements. However, this potential is shown to be limited due to the reduced discerning capability between physical and regularizing terms, particularly as the model updating algorithm converges. A simple numerical case study allowed the concepts to be presented visually.

The analysis and concepts in this study may be extended to more demanding, but structurally richer, scenarios of model-based inversion or updating of over-parameterized response models.
References


