Constrained Generic Substructure Transformations

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Abstract

Generic substructure transformations are examined. Through consideration of the desired properties of
the transformed substructure, a series of constraint equations are identified; the zero stiffness constraint
is demonstrated which enforces the connectivity present in the baseline substructure. The transformation
matrix is included implicitly in the optimisation problem defined by the constraint equations. The solution
to the optimisation will yield a member of the generic family of substructures that satisfies the constraint
equations. A transformation has been applied to a simple substructure to demonstrate the method.

1 Introduction

A finite element model is a numerical approximation of the mass and stiffness properties of a structure. Each
element in the model represents a small part of the structure. The behaviour of this small part is approximated
to that of one of the standard element types: a beam, a plate, a shell etc. There will be areas of uncertainty
within any modern aerospace structure where additional behavioural effects come into play; for example
stress stiffening around a bolted joint. It is in areas such as this that the standard finite element model will
break down.

A clear methodology has been adopted to tackle this problem. The areas of uncertainty must first be iden-
tified. A parameter(s) can then be introduced to effect the desired change to the model mass and stiffness
properties. Finally the parameter(s) value is updated using the sensitivity method to gain an improved esti-
mate. Examples of joint parameterisation are

- Translational and rotational spring model of a welded joint [1].
- Rigid offsets in beam elements to account for the stiffening effect at a joint [2].
- Micro slip conditions in the modelling of bolted joints [3].

Parameterising the error in the baseline model enables the changes that are made during updating to be jus-
tified physically. If physical meaning is lost then the model is useless, regardless of whether it reproduces
the experimental data or not (this is why direct methods in finite element model updating are no longer
promoted). Although the new parameter can be justified physically, it is highly unlikely that it will ade-
quately approximate the multitude of dynamic interactions that occur in a real joint. More parameters can be
introduced but it will never be possible to model every aspect of the dynamics.
Gladwell and Ahmadian [4] proposed the use of generic elements in model updating. A generic element is a member of a generic family of elements; e.g. rod elements, beam elements, shell elements etc. The generic family of elements can be defined which encompasses all possible element formulations within the family. A family of 2D beam elements would include all elements whose behaviour can be described by the standard six degrees of freedom i.e lumped mass Euler-Bernoulli, distributed mass Timoshenko, and everything else beyond and inbetween. The baseline element for the family is defined as that with the simplest formulation i.e. lumped mass, Euler Bernoulli. Through a transformation of the baseline eigenvectors and eigenvalues it is possible to reproduce the matrices for any member of the family. As a result any theoretical beam model can be obtained without the need to explicitly parameterise the additional behavioural effects.

The generic element approach can be applied to substructures in the modelling of joints. A baseline joint element is required to define the generic family. This baseline joint is assembled using a number of standard elements; for example beam elements (the type of element will depend on the specific structural joint that is being modelled). Once defined, the baseline joint element eigendata can be modified to obtain mass and stiffness matrices for a different member of the generic family of joint elements. The degree by which the eigendata is modified can be used as parameters in an updating scheme. This will identify the member of the generic family that matches the experimental data most accurately. The first two eigenvalues of a T-joint substructure element were included as parameters in the updating of a welded frame [5]. A high level of correlation was achieved with regard to both natural frequencies and mode shapes: i.e. maximum 2% frequency error and a minimum MAC of 97%.

To ensure that physical meaning is maintained it is important to examine the properties of the updated generic element. In the more traditional approach to model updating the parameters are physically identifiable and often represent geometrical quantities (in which case tolerance bands can be used enforce physical meaning). The meaning of the stiffness changes that occur through a generic element transformation cannot be directly interpreted from the changes to the baseline eigendata. The purpose of this paper is to investigate generic element transformations, and to assess whether any control can be exerted over the type of stiffness changes that occur.

2 Generic elements

2.1 Generic element stiffness matrix decomposition

The generic element matrices are defined through the decomposition of the baseline matrices. A full derivation can be found in [4]. At this point it will be assumed that model error is restricted to the stiffness matrix. For this reason a derivation of the generic element mass matrix is not presented here. The eigenvalue problem is given by

\[ [\Phi_0]^T [M_0] [\Phi_0] = [I] \]
\[ [\Phi_0]^T [K_0] [\Phi_0] = [\Gamma_0] \]

where \([\Phi_0]\) is the mass normalised eigenvectors matrix, \([M_0]\) and \([K_0]\) are the mass and stiffness matrices and \([\Gamma_0]\) is the diagonal matrix of eigenvalues. The subscript 0 identifies that these are the matrices for the baseline element. Eigenvectors of a different member of the generic family of elements are related to the baseline eigenvectors by the transformation matrix \([S]\). The subscript \(T\) is introduced to identify the matrices of the transformed element. Thus,

\[ [K_T] = [\Phi_T]^T [\Gamma_T] [\Phi_T]^{-1} \]
\[ [\Phi_T] = [\Phi_0][S]^{-1} \]
The transformation matrix is divided into rigid body and strain modes; the subscript \( s \) is introduced to denote strain modes. Only strain modes are of interest when considering the stiffness matrix. Equations (1), (2), (3) and (4) can be manipulated to obtain the generic element stiffness matrix as

\[
[K_T] = [M_0][\Phi_0s][V][\Phi_0s]^T[M_0]
\]

where,

\[
[V] = [S_s]^T[\Gamma_T s][S_s]
\]

By varying the elements of \([V]\), the matrices for different members of the generic family will be obtained. The form of \([S_s]\) can be defined to retain properties of the baseline element. If \([S_s]\) is defined such that the transformed modes are linear combinations of baseline modes which exhibit the same symmetry, then that symmetry will be retained in \([K_T]\).

### 2.2 Generic substructures

The baseline generic substructures considered from this point onward will be constructed from lumped mass, Euler-Bernoulli beam elements. The lumped mass model will give a diagonal mass matrix. This simplifies the cases examined (mass terms in equation (7) are scalar quantities), but does not restrict the use of the method presented with a more complex mass model. \([\Gamma_T s]\) is also diagonal. The vector \([\phi_{0s}]\) is defined as the \( s \)th row of \([\Phi_0s]\), and is a vector of the nodal displacements for each strain mode at degree of freedom \( i \). Element \((i,j)\) of the generic element stiffness matrix can therefore be expressed as

\[
[K_T]_{ij} = [M_0]_{ii}[M_0]_{jj}[\Phi_0s][S_s]^T[\Gamma_T s][S_s][\phi_{0s}]^T
\]

It is necessary to consider an example to examine the implications of equations (7). The stiffness matrix and eigenvectors of a 2D, lumped mass, Euler-Bernoulli beam element are given by equations (8) and (9).

\[
[\Phi_0^e] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1/2 \\
0 & 2\sqrt{3} & 3 \\
-1 & 0 & 0 \\
0 & 0 & -1/2 \\
0 & -2\sqrt{3} & 3
\end{bmatrix}
\]

\[
[K_0^e] = k_0 \begin{bmatrix}
k_a & 0 & 0 & -k_a & 0 & 0 \\
0 & 12 & 6 & 0 & -12 & 6 \\
k_a & 0 & 0 & -k_a & 0 & 0 \\
0 & 6 & 4 & 0 & -6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & -6 & 0 & 12 & -6 \\
0 & 6 & 2 & 0 & -6 & 4
\end{bmatrix}
\]

As this is a beam element rather than a substructure, the superscript \( e \) is introduced for clarity. The axial and transverse motion of the beam element is uncoupled. Elements of the stiffness matrix that represent the constraining stiffness between uncoupled degrees of freedom are therefore equal to zero in the baseline matrix; for example \([K_0]_{1,2} = 0\). The corresponding rows of \([\Phi_{0s}]\) that contribute to \([K_0]_{1,2}\) are

\[
[\phi_{0s1}] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\phi_{0s2}] = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}
\]

\[
(10)
\]
The baseline modes also exhibit symmetry, and to preserve this through the transformation, \([S_s]\) must have the form

\[
[S_s] = \begin{bmatrix}
s_1 & 0 & 0 \\
0 & s_2 & 0 \\
0 & 0 & s_3 \\
\end{bmatrix}
\]  

(11)

If degree of freedom \(i\) and degree of freedom \(j\) are uncoupled then

\[
[\phi_{0si}][S_s]^T[\Gamma_{Ts}][S_s][\phi_{0sj}]^T = 0
\]

(12)

for all values of \([S_s]\) and \([\Gamma_{Ts}]\) (providing \([S_s]\) is of the form given in equation (11)); hence the zeros in \([K_0]\) will be preserved through a transformation to \([K_T]\).

This is not the case for generic substructures. These are assembled from two or more beam elements, and it is therefore likely that the local axes of the individual beams will not be aligned, causing coupling between global axial and transverse degrees of freedom. There will also be nodes within the substructure that are not directly connected to one another. This means that degrees of freedom which are uncoupled or unconnected may be active in the same mode.

\[
[\phi_{0si}][S_s]^T[\Gamma_{Ts}][S_s][\phi_{0sj}]^T = 0
\]

(13)

is now only guaranteed to hold when

\[
[S_s] = I, \quad \text{and} \quad [\Gamma_{Ts}] = [\Gamma_{0s}] \implies [K_T] = [K_0]
\]

(14)

Hence it is highly likely that the zeros in \([K_0]\) will be lost through a transformation to \([K_T]\) (where \([K_T] \neq [K_0]\)). Whether or not the zeros should be preserved is an important question. It seems sensible that degrees of freedom with no physical connection should have no constraining stiffness between them. In other instances the introduction of new non-zero terms in \([K_T]\) may help to capture the interactions that take place on a real structure between seemingly uncoupled degrees of freedom. If the desired zeros are to be preserved then the transformation must be constrained.

### 3 Zero stiffness constraint

If equation (13) is satisfied then the zeros will be preserved through a transformation from \([K_0]\) to \([K_T]\); thus

\[
[K_T]_{ij} = 0, \quad \text{when} \quad [K_0]_{ij} = 0
\]

(15)

It is assumed that there exists a strain mode transformation matrix that will enforce equation (15) for a given \([\Gamma_{Ts}]\). No assumptions are made as to the form of \([S_s]\). The modified eigenvector matrix is defined as

\[
[\Phi_{Ms}] = [\Phi_{0s}][S_s]^T
\]

(16)

The transformation matrix is now included within the modified eigenvector matrix. This simplifies equation (7) and enables it to be expressed as the summation

\[
[K_T]_{ij} = [M_0]_{ii}[M_0]_{jj} \sum_{p \leq m} [\Gamma_{Ts}]_{pp}[\Phi_{Ms}]_{ip}[\Phi_{Ms}]_{jp}
\]

(17)
where \( p \) is the strain mode index, and \( m \) is the total number of strain modes. A constraint equation can be defined for each zero that is to be preserved in \([K_T]\), using equation (17). The constraint equations can be used to define a non-linear optimisation problem, the parameters of which are the elements of \([\Phi_M]\). If a solution can be found, \([\Phi_M]\) can be used to calculate the transformed stiffness matrix using

\[
[K_T] = [M_0][\Phi_M][\Gamma_T][\Phi_M]^T[M_0]
\]  

(18)

If \( r \) is the number of rigid body modes, and \( n_m \) the number of flexible modes, then the number of parameters \((n_{par})\) and the number of constraints \((n_{con})\) can be expressed as

\[
n_{par} = m(m + r), \text{ and } n_{con} < (m + r)(m + r)/2
\]  

(19)

The number of parameters in the optimisation will exceed the number of constraint equations when \( m \geq r \); hence when applied to substructures, the problem will be under-determined and an infinity of solutions will exist. Each solution will produce a transformation to a different member of the generic family of substructures. Although each will satisfy the zero stiffness constraints, it is impossible to attach physical meaning to the changes that occur during the transformation. The particular member of the family that the optimisation yields will essentially be random, and will be heavily reliant on the basis and the optimisation technique employed.

It is intuitive that the stiffness changes made during the transformation should not radically alter the substructure, but rather fine tune it to account for the modelling discrepancies. A good basis for the optimisation is therefore the baseline eigenvectors. Properties of the baseline eigenvectors can also be inferred onto the modified eigenvectors; for example symmetry considerations. By manipulating the parameter set in this way it is possible to direct the optimisation towards members of the generic family with the desired properties. In practical terms this is achieved by setting parameters to equal one another (symmetry), by setting parameters equal to zero (degree of freedom inactive for given mode) or by setting parameters equal to the un-modified baseline value (mode remains unchanged). In mathematical terms these are additional constraints, and if a sufficient number are defined a unique solution can be found.

Equation (16) will be undetermined. The modified eigenvectors are calculated outside the constraints of this equation, so there is no guarantee that an exact solution will exist. An approximate solution can be found using the Moore-Penrose pseudo inverse to give

\[
[S_s]^T = ([\Phi_0s]^T[\Phi_0s])^{-1}[\Phi_0s]^T[\Phi_M]
\]  

(20)

If the solution is too inaccurate, then the transformation cannot be represented by \([S_s]\) and physical meaning will have been lost. The parameter restrictions that are introduced will effect the properties of equation (20) and it may be possible to manipulate them to improve the accuracy of the solution. An investigation into this has not been undertaken here, but will be required before full confidence in the method can be gained.

4 Case study

4.1 Substructure transformation

Consider the baseline generic substructure shown in figure 1 constructed from two beam elements. The baseline stiffness matrix is also given. It is colour coded; the white cells depict zero terms and the shaded cells non-zero terms.

It is assumed that all zeros are to be preserved through a transformation. Constraint equations are generated using equation (17). For example, there is to be no constraining stiffness between degree of freedom 1 and degree of freedom 9. Setting \([K_{gen}]_{0,1} = 0\) will yield the constraint equation
The baseline substructure shown in figure 1 has 44 zeros and hence 22 potential constraint equations. It has 9 degrees of freedom and 6 flexible modes. This makes a total of 54 parameters for the 22 constraint equations.

The mode shapes for the substructure (not to scale) are given in figure 2. By identifying symmetry, anti-symmetry and inactive degree’s of freedom within the modes, the number of parameters can be reduced. Baseline flexible mode 1 will be discussed as an example. It exhibits symmetry and anti-symmetry allowing degrees of freedom 5, 7, 8 and 9 to be expressed in terms of degrees of freedom 4, 2, 1 and 3 respectively. There is also no rotation of node 2, which enables degree of freedom 6 to be set to zero. Through similar considerations of all strain modes, \([\Phi_0]\) can be expressed as

\[
[\Phi_0] = \begin{bmatrix}
\phi_{011} & 0 & \phi_{013} & \phi_{014} & \phi_{015} & \phi_{016} \\
\phi_{021} & 0 & \phi_{023} & \phi_{024} & \phi_{025} & \phi_{026} \\
\phi_{031} & \phi_{032} & \phi_{033} & \phi_{034} & \phi_{035} & \phi_{036} \\
\phi_{041} & 0 & -\phi_{043} & \phi_{044} & \phi_{045} & \phi_{046} \\
-\phi_{041} & 0 & \phi_{043} & -\phi_{044} & \phi_{045} & -\phi_{046} \\
0 & \phi_{062} & 0 & \phi_{034} & 0 & \phi_{036} \\
\phi_{021} & 0 & \phi_{023} & -\phi_{024} & \phi_{025} & -\phi_{026} \\
\phi_{011} & 0 & \phi_{013} & -\phi_{014} & -\phi_{015} & -\phi_{016} \\
-\phi_{031} & \phi_{032} & -\phi_{033} & \phi_{034} & -\phi_{035} & \phi_{036}
\end{bmatrix}
\]  

If these restrictions are inferred onto the modified eigenvectors, the number of parameters in the optimisation is reduced to 22; hence a unique solution can be found.

The arbitrary eigenvalue transformation given by

\[
\gamma_{T_F} M_{11} \phi_{M91} + \gamma_{T2} M_{12} \phi_{M92} + \gamma_{T3} M_{13} \phi_{M93} + \gamma_{T4} M_{14} \phi_{M94} + \gamma_{T5} M_{15} \phi_{M95} + \gamma_{T6} M_{16} \phi_{M96} = 0
\]  

where,

\[
\gamma_{TP} = [\Gamma_T]_{pp}, \text{ and } \phi_{Mij} = [\Phi_M]_{ij}
\]  

The arbitrary eigenvalue transformation given by
is applied to the substructure in figure 1, under the eigenvector constraints defined by equation (23). Figure 3 summarises the percentage changes to the magnitude of the elements of $[K_0]$ through the transformation to $[K_T]$. Each percentage change is related to the degree(s) of freedom to which it applies. The off-diagonal terms are given for the beam element defined by nodes 1 and 2 only due to symmetry. The transformation used (equation (24)) represents an increase in the eigenvalues of the first and second modes. These are both bending modes, and as a result the changes to the axial elements of $[K_0]$ are negligible and have been omitted from figure 3.

Generally a stiffer structure will have higher natural frequencies. The general increase in all diagonal terms corresponds to the increase in eigenvalues. The translational stiffnesses at node 2 are dominated by the axial stiffness of the beams and hence show no significant change. Off-diagonal terms in the stiffness matrix represent the constraining stiffness between two degrees of freedom; i.e. $[K]_{2,5}$ is the constraining stiffness between transverse translation of node 1 and transverse translation of node 2. A general increase in magnitude of these terms is also exhibited through the transformation, with the exception of the stiffness constraint between the rotational degrees of freedom.

The actual values of the changes made to the elements of the transformed stiffness matrix are not critical in this study. $[\Gamma_{TS}]$ has been predefined, the structure is purely theoretical and so $[K_T]$ doesn’t mean anything physically. What is important, is the type of changes that have occurred through the transformation, and how
they can be related to types of behaviour that may occur in a real structure. As already stated, the effect of the transformation on the axial modes is minimal and the substructure element has a line of symmetry. Transverse motion of beam 1 only will therefore be considered.

4.2 Shape function considerations

The baseline structure is composed of two lumped mass, Euler-Bernoulli elements. The element stiffness matrix is derived using the standard finite element formulation:

$$\left[K_{\text{beam}}\right]_{ij} = \int [B]^T \left[E\right] [B]_{ij} dV$$

(25)

where $[B]$ is the strain-displacement matrix and $[E]$ is the elasticity matrix. For transverse deflections cubic shape functions are employed, and $[B]$ is the second derivative.

The objective of this section is to determine the locations within the element that are affected by the change in the stiffness matrix. This will be demonstrated by suitable changes to the shape functions. It may be thought that the shape functions will not change, although the shape functions for Euler-Bernoulli and Timoshenko beam elements are different. If the differences in the stiffness matrix arise from changes in the stiffness within the element then this will also be revealed in the estimated shape functions since the integration in equation (25) combines the local stiffness with the shape functions. Of course only an averaged local stiffness variation may be estimated based on the deformations due to the shape functions.

The magnitude of many terms within the elemental stiffness matrix are equal. Equation (9) shows that

$$[K_{0}^e]_{2,3} = [K_{0}^e]_{2,6} = -[K_{0}^e]_{5,3} = -[K_{0}^e]_{5,6}$$

(26)

If the beam is uniform, and the length of the beam is normalised using $\xi = x/L$:

$$\int_{0}^{1} B_{2}^T \frac{\partial B_{3}}{\partial \xi} d\xi = \int_{0}^{1} B_{2}^T \frac{\partial B_{6}}{\partial \xi} d\xi = -\int_{0}^{1} B_{5}^T \frac{\partial B_{3}}{\partial \xi} d\xi = -\int_{0}^{1} B_{5}^T \frac{\partial B_{6}}{\partial \xi} d\xi$$

(27)
Each of these terms represents the stiffness constraint linking a translational and a rotational degree of freedom within beam 1. The contribution of each degree of freedom (of a given type) to the overall deflection of the beam is equal; hence a symmetric set of nodal displacement will yield a symmetric/anti-symmetric beam deflection. This is clearly directly related to the shape functions used.

\[ N_2(\xi) = 1 - N_5(\xi), \quad \rightarrow \quad B_2(\xi) = -B_5(\xi) \]  
\[ N_3(\xi) = -N_6(1 - \xi) \quad \rightarrow \quad \frac{dB_3(\xi)}{d\xi} = \frac{dB_6(\xi)}{d\xi} \]

The relations defined in equation (26) no longer hold when applied to the transformed element. The magnitude of the stiffness constraints between degree of freedom 2 and rotational degrees of freedom are now greater than those between degree of freedom 5 and the rotational degrees of freedom.

\[ ||[K_T^{\xi}]_{2,3}|| > ||[K_T^{\xi}]_{5,3}|| = ||[K_T]_{5,6}|| \]

\[ \int_0^1 B_2 B_3 d\xi = \int_0^1 B_2 B_6 d\xi > \int_0^1 B_3 B_5 d\xi = \int_0^1 B_3 B_6 d\xi \]

It is useful to relate these new relationships to possible changes to the baseline element shape functions. An exact definition of the new shape function cannot be made from the information available. Insight into the types of change that occur through the transformation can however be gained. The general strain displacement equation for transverse deflection is the linear function in \( \xi \)

\[ B_i = a_i + b_i \xi \]

At this point the subscripts 0 and T are introduced to distinguish between baseline and transformed shape functions, and strain displacement functions. \( B_{02} \) and \( B_{05} \) are straight lines for which \( B_{0i} = 0 \) at the beam centre line (the turning point of the corresponding shape function). This is expressed mathematically as:

\[ a_{0i} = -\frac{b_{0i}}{2} \]

If equation (33) holds for the transformed strain displacement functions, equation (25) can be used to show that:

\[ [K_T^{\xi}]_{2,3} = A \frac{b_{T2} b_{T3}}{12}, \quad [K_T^{\xi}]_{2,6} = A \frac{b_{T2} b_{T6}}{12}, \quad [K_T^{\xi}]_{5,3} = A \frac{b_{T5} b_{T3}}{12}, \quad [K_T^{\xi}]_{5,6} = A \frac{b_{T5} b_{T6}}{12} \]

\[ \Rightarrow b_{T3} = b_{T6} \]

where \( A = \frac{EI}{L^3} \).

Hence the transformed strain displacement functions for degrees of freedom 2 and 5 are scalar multiples of the baseline functions:

\[ B_{T2} = \eta_2 B_{02}, \quad B_{T5} = \eta_5 B_{05} = -\eta_5 B_{02} \]

where \( \eta_2 \) and \( \eta_5 \) are constants and \( \eta_2 > \eta_5 \). This line of reasoning is further backed up through consideration of the stiffness constraints between the translational degrees of freedom within the element. Inspection of figure 3 gives:
Through integration of the strain displacement functions and application of suitable boundary conditions, it is possible to identify the trends in the changes to the shape function with $\eta$, see figure 4. In this study $\eta_2 > \eta_5$, and as a result the transformed shape functions will no longer give a symmetrical stiffness distribution along each beam within the substructure element. This could be applied to areas of uncertainty where there will be additional behavioural effects, such as local stiffening around a joint.

![Figure 4: Shape function trends through transformation](image)

5 Discussion

In the traditional approach to generic element modelling, a transformation from $[K_0]$ to $[K_T]$ is applied by modifying the elements of $[V]$ where

$$[V] = [S_s]^T [\Gamma_{Ts}] [S_s]$$

The form of $[S_s]$ can be manipulated to preserve certain properties of the baseline element; for example symmetry within the modes. It has been demonstrated that when applied to substructure elements, this method is likely to cause the zeros in $[K_0]$ to be lost through a transformation to $[K_T]$.

An alternative approach to generic element modelling has been presented in this paper. A series of constraint equations have been generated which enforce the preservation of all zeros in $[K_0]$ through a transformation. It has been assumed that for a given $[\Gamma_{Ts}]$ there exists a transformation matrix that will satisfy the constraint equations. Rather than identifying $[S_s]$ directly, it has been incorporated into the modified eigenvector matrix.

$$[\Phi_{Ms}] = [\Phi_{0s}] [S_s]^T$$
The elements of $[\Phi_M]$ are parameters in the non-linear optimisation problem defined by the constraint equations. The problem will be under-determined and in order to find a unique solution, a set of restrictions are placed on the form of $[\Phi_M]$ to reduce the number of parameters. Each possible set will infer different properties onto $[K_T]$ (and will hence produce a different member of the generic family) but at the same time will preserve the zeros in $[K_0]$. This approach to generic element modelling can be separated into three distinct stages: the definition of the constraint equations, the application of parameter restrictions and the transformation of the eigenvalues.

Each constraint equation is generated through consideration of a specific element or elements of $[K_T]$; the zero stiffness constraints were defined using $[K_T]_{ij} = 0$. As well as the zero stiffness constraints there are a number of other constraints that can potentially be applied. In section 4.2, a non-symmetric stiffness distribution within the beam elements of the transformed substructure was identified. If a symmetric stiffness distribution is desired between certain degrees of freedom then a constraint can be defined of the form

$$[[K_T]_{ij}] = [[K_T]_{kl}]$$

(41)

where an expression for $[K_T]_{ij}$ is obtained using equation (17).

It has been assumed in the examples given that every zero in the baseline stiffness matrix must be preserved. The introduction of new non-zero terms within $[K_T]$ may be useful in modelling an interaction between degrees of freedom that has not been accounted for in $[K_0]$. Non-zero terms can be allowed to occur naturally by removing zero stiffness constraints, or can be introduced artificially through modification of the zero stiffness constraint equation

$$[K_T]_{ij} = \alpha$$

(42)

where $\alpha$ can be included as a parameter in an updating scheme.

There is a great deal of choice when defining the parameter set restrictions; it has been shown that for a substructure composed of two beam elements, a reduction in the number of parameters from 54 to 22 was required. A relationship between $[\Phi_M]$ and $[\Phi_T]$ cannot be identified directly without consideration of the rigid body transformation matrix. This makes it more difficult to interpret the effect of the chosen parameter restrictions.

The elements of $[K_T]$ are calculated using equation (17). This equations sums the influence of the relevant degree’s of freedom over all modes. If the baseline modes exhibit symmetry, then that symmetry can be enforced upon $[\Phi_M]$, and so the influence of degree’s of freedom across the line of symmetry will be equal. This symmetry will be reflected in the elements of $[K_T]$ and hence must also be inferred onto $[\Phi_T]$.

Inspection of the baseline eigenvectors reveals that there are degrees of freedom which are inactive in certain modes. If $[\Phi_0]_{ij} = 0$ then degree of freedom $i$ is inactive in mode $j$. Substituting the baseline element matrices into equation (17) gives

$$[K_0]_{ij} = [M_0]_{ii}[M_0]_{jj} \sum_{p \leq m} [\Gamma_0]_{pp}[\Phi_0]_{ip}[\Phi_0]_{jp}$$

(43)

This means that eigenvalue $j$ of the baseline element has no influence on the stiffness constraint between degree of freedom $i$ and any other degree of freedom. Setting the corresponding degrees of freedom in $[\Phi_M]$ to zero will restrict the influence of the transformed element eigenvalues to follow those of the baseline element. For example if $[\Phi_M]_{21}$ is set equal to zero, then a transformation of eigenvalue 1 only (from $\gamma_{0,1}$ to $\gamma_{T,1}$) will have no direct influence on the stiffness constraints involving degree of freedom 2. This is not to say that these stiffness constraints will go unchanged, as the eigenvector elements for this degree of freedom in other modes are likely to be modified from the baseline value to satisfy the zero stiffness constraint.
When using a transformation as part of an updating scheme it is the eigenvalues of a small number of the lower modes that will generally be modified. Parameters can be removed from the optimisation by fixing $[\Phi_{Ms}]_{ij}$ equal to $[\Phi_{0s}]_{ij}$. Entire modes can be removed by applying this restriction to columns of $[\Phi_{Ms}]$. This can be used to concentrate the differences between $[\Phi_{Ms}]$ and $[\Phi_{0s}]$ to the modes of interest. This type of restriction will occur naturally to some extent. Most optimisation techniques utilise a gradient search to identify those parameters to which the constraint equations are most sensitive. This will effectively select the degrees of freedom within the modes whose eigenvalues are being modified, and so changes to these parameters will be more pronounced.

The optimisation problem is defined uniquely by the chosen constraint equations and parameter restrictions. Certain properties are inferred onto the transformed element, some directly and some indirectly, but the specifics of the transformed element cannot be assessed without $[\Gamma_{Ts}]$. It can be visualised that a given optimisation defines a sub-group of elements with similar properties within the generic family. Different members of the sub-group will be obtained as the transformed eigenvalues are varied. When the elements of $[\Gamma_{Ts}]$ are used as parameters in an updating scheme, the final updated element will be a member of the sub-group of elements defined by the optimisation problem. Ensuring that the sub-group contains elements with properties that suitably represent the real structure is therefore paramount. There are so many combinations of constraints and parameter restrictions that it is not possible to define a general optimisation problem that can be applied to any substructure element. A unique optimisation problem must be defined in each case.

The properties that the constraint equations introduce can be interpreted easily and so these should be decided upon first. The relevant zeros (or properties from alternative constraints) in $[K_0]$ are then guaranteed to be preserved through the transformation. Before considering the parameter restrictions it is important to gauge an understanding of the baseline element. The substructure element will be a small part of a larger model. It is therefore necessary to identify which substructure modes are active in the global modes of interest (they are likely to be the lower modes). Modifying the eigenvalues of these modes will have the most pronounced impact on the global dynamics, and so should be chosen as the updating parameters. The restrictions to the parameter set can now be considered. This is best done through experimentation with a number of different potential sets through predefined transformations. An idea of the properties of the sub-group of elements for each set of restrictions can be gained, and the most suitable optimisation problem can be chosen.

At each iteration in the updating scheme a transformation is applied, and hence an optimisation problem must also be solved. If a joint element is being examined, then the updating will be computational expensive for large structures which contain a number of joints. If a given type of joint can be isolated and updated, then it should be possible to identify the changes to the element, and to parameterise the properties, resulting in a parameterised joint element of a given type.

Acknowledgements

The authors would like to thank EPSRC and Westland Helicopters for supporting the work presented in this paper.

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