STRUCTURE PRESERVING TRANSFORMATIONS AND ISOSPECTRAL FLOWS
FOR SECOND ORDER SYSTEMS

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ABSTRACT
When the dynamics of any general second order system are cast in a state-space format, the initial choice of the state-vector usually comprises one partition representing system displacements and another representing system velocities. Coordinate transformations can be defined which result in more general definitions of the state-vector. This paper discusses the general case of coordinate transformations of state-space representations for second order systems. It identifies one extremely important subset of such coordinate transformations - namely the set of structure-preserving transformations for second order systems - and it highlights the importance of these. It shows that one particular structure-preserving transformation results in a new system characterised by real diagonal matrices and presents a forceful case that this structure-preserving transformation should be considered to be the fundamental definition for the characteristic behaviour of general second order systems - in preference to the eigenvalue-eigenvector solutions conventionally accepted.

The regular $\lambda$-matrix $\lambda^2 M + \lambda D + K$ with $M, D, K \in \mathbb{R}^{n \times n}$ defines a second-order system. A one-parameter trajectory of such a system $\{M(t), D(t), K(t)\}$ is an isospectral flow (or more correctly an equivalence flow) if the eigenvalues and the dimensions of the associated eigenspaces are the same for all parameter values $t \in \mathbb{R}$. This paper presents the general form for real isospectral flows of real-valued second order systems.

INTRODUCTION
Consider the $N$ degree of freedom second order system characterised by the real $(n \times n)$ matrices $\{K, D, M\}$ and having displacement vector $q$ and force vector $Q$. Then the equation of motion is

$$M\ddot{q} + D\dot{q} + Kq = Q$$  \hspace{1cm} (1)

where $\dot{q}$ and $\ddot{q}$ represent the first and second derivatives of the vector $q$ with respect to time.

This paper is concerned with transformations to express this general system in different but equivalent forms. Because system matrices, $\{K, D, M\}$, are not always symmetric, or self-adjoint, different transformations may be applied to the left and right of these matrices. Self-adjoint systems with symmetric matrices are a special case, and the left and right transformations will usually be identical so that symmetry is preserved after the transformation. Garvey et al. [1] gave more details.

Consider the matrix pair $\{K, M\}$ and the associated eigenvectors. Denoting the diagonal matrix of eigenvalues as $\Lambda$ and the corresponding matrices of left and right eigenvectors as $\{\Phi_L, \Phi_R\}$ respectively. If all of the eigenvalues are distinct, it is simple to show that with appropriate scaling of the eigenvectors,

$$\Phi_L^T K \Phi_R = \Lambda, \quad \Phi_L^T M \Phi_R = I$$  \hspace{1cm} (2)

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In some cases where the eigenvectors are not distinct, it is not possible to find a full-rank \((n \times n)\) matrix \(\Phi\) satisfying equation (2). Such systems are referred to as defective systems. The other exception occurs when \(M\) is singular. Although this paper will not consider these systems in depth, the techniques described readily extend to such systems.

A system is described as classically or proportionally damped if the same transformation matrices \(\{\Phi_L, \Phi_R\}\) that diagonalises the mass and stiffness matrices also diagonalises the damping matrix. Caughey and O’Kelly \([2]\) discuss this in the context of self-adjoint systems, although this may be extended to the general case \([1]\). If general scaling of the eigenvectors is considered then

\[
\Phi_L^T K \Phi_R = K_D, \quad \Phi_L^T D \Phi_R = D_D, \quad \Phi_L^T M \Phi_R = M_D \tag{3}
\]

where \(\{K_D, D_D, M_D\}\) are diagonal matrices. Together \(\{\Phi_L, \Phi_R\}\) describe a transformation from the original set of displacement coordinates, \(q\), and its corresponding vector of forces, \(Q\), to a new set of displacement coordinates \(r\) and the corresponding vector of forces \(R\) through

\[
q = \Phi_R r, \quad R = \Phi_L^T Q. \tag{4}
\]

Then, the original equation of motion for a classically damped system is transformed to

\[
M_D \ddot{r} + D_D \dot{r} + K_D r = R \tag{5}
\]

Because the equations in (5) are completely decoupled, the combination of equations (4) and (5) provides for the very efficient calculation of response in the time or frequency domains through the use of superposition. It also provides for a clear understanding of the mechanisms through which the system responds (especially when \(\{\Phi_L, \Phi_R\}\) are real). The left modal matrix, \(\Phi_L\), acts to transform physical forces into corresponding modal forces and the right modal matrix, \(\Phi_R\), acts to transform modal displacements into physical displacements.

For systems that are not classically damped, the situation is not nearly so clear using present-day methods. In general, there is no pair of \((n \times n)\) matrices \(\{\Phi_L, \Phi_R\}\) (real or complex) that can simultaneously diagonalise the three system matrices according to equation (3). The original system can be represented as a system of first order differential equations in state-space form. In this case, the two system matrices in the state-space equation each have dimension \((2n \times 2n)\) and the inherent second order nature of the original system is effectively ignored. The \(2n\) characteristic roots and their associated \(2n\) modal vectors (left and right) can be computed but in general these are complex and their full significance is difficult to grasp \([3]\). Many researchers have battled with the implications of complex modes in various contexts including interpretation of complex modes, the search for iterative or approximate solutions for the damped natural frequencies and for system response, model correlation, model updating and system identification and model reduction of generally-damped systems.

The first priority of this paper is to show that real-valued transformations do exist for most real second order systems such that system response can be assembled as the direct sum of contributions from \(n\) decoupled single degree of freedom second order systems. These transformations exist for all real second order systems having no repeated pairs of characteristic roots and they are referred to here as diagonalising structure-preserving transformations. One major application is considered, namely the general form of isospectral flows for second order systems. This paper only considers second order systems, but the proposed techniques are readily extended to higher order systems.

**STRUCTURE PRESERVING TRANSFORMATIONS**

Real-valued structure preserving transformations (SPTs) have been introduced and studied recently by Garvey et al. \([1]\) for second order systems. The transformations are based on the state space representations of the equations of motion. For a second order system there are three possible ways of writing the state space representation in terms of the displacement and velocity, ensuring that symmetry is preserved if the mass, damping and stiffness matrices are symmetric. Thus, the equations of motion (1) may be written as,

\[
- \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 & K_0 \\ K_0 & D_0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ Q \end{bmatrix} \tag{6}
\]

\[
\begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} Q \\ 0 \end{bmatrix} \tag{7}
\]

\[
\begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 & K_0 \\ K_0 & D_0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix} \tag{8}
\]

where

\[
p = \dot{q}, \quad P = \dot{Q}. \tag{9}
\]

Note that the 0 subscript has been applied to show that the matrices are those before transformation.

Only three different matrices, denoted \(A_0, B_0, C_0 \in \mathbb{R}^{2n \times 2n}\), are used in equations (6)–(8), and these are defined as

\[
A_0 := \begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix} \tag{10}
\]

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\[ B_0 := \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix} \quad (11) \]
\[ C_0 := \begin{bmatrix} 0 & K_0 \\ K_0 & D_0 \end{bmatrix} \quad (12) \]

Lancaster [4] considered the analysis of linear systems of arbitrary order, and introduced special forms for the state space representation of these systems. Matrices \( A_0, B_0, C_0 \) are consistent with Lancaster’s definitions [4] and the matrices \( A_0, B_0, C_0 \) will be called a real Lancaster triple.

**Definition: Structure Preserving Transformations.** Any pair of matrices \( T_R, T_L \in \mathbb{R}^{2n \times 2n} \) is called a structure preserving transformation if for some real Lancaster triple \( A_0, B_0, C_0 \), the three matrices

\[ A := T_L^T A_0 T_R \quad (13) \]
\[ B := T_L^T B_0 T_R \quad (14) \]
\[ C := T_L^T C_0 T_R \quad (15) \]

also form a real Lancaster triple. This means that the transformation retains the form for the state space matrices, in the sense that the zero sub-matrices are retained, and the separate occurrences of the mass, damping and stiffness matrices in \( A, B, \) and \( C \) are all equal.

Where \( M_0 \) is non-singular, structure-preserving transformations are fully defined by the preservation of form in \((T_L^T A_0 T_R), (T_L^T B_0 T_R)\) and the system eigenvalues can be obtained by solving the generalised eigenvalue problem. In this case it is easy to demonstrate that structure is preserved also in \((T_L^T C_0 T_R)\) since

\[ C_0 = -B_0 A_0^{-1} B_0 \quad (16) \]

Note that non-singular \( M_0 \) implies that \( A_0 \) is non-singular [5]. Where \( K_0 \) is non-singular, \( C_0 \) must also be non-singular and in this case, structure-preserving transformations can be defined fully by the preservation of form in \((T_L^T B_0 T_R), (T_L^T C_0 T_R)\). In this case structure is invariably also preserved in \((T_L^T A_0 T_R)\) since

\[ A_0 = -B_0 C_0^{-1} B_0 \quad (17) \]

To consider the form of the constraints for SPTs, suppose

\[ T_L := \begin{bmatrix} W_L & X_L \\ Y_L & Z_L \end{bmatrix} \quad (18) \]
\[ T_R := \begin{bmatrix} W_R & X_R \\ Y_R & Z_R \end{bmatrix} \quad (19) \]

The constraints for the system to be structure preserving are given explicitly by

\[ \begin{bmatrix} W_L X_L^T \\ Y_L Z_L^T \end{bmatrix} D_0 M_0 \begin{bmatrix} W_R X_R \\ Y_R Z_R \end{bmatrix} = \begin{bmatrix} D \quad M \\ M \quad 0 \end{bmatrix} \quad (20) \]
\[ \begin{bmatrix} W_L X_L^T \\ Y_L Z_L^T \end{bmatrix} \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix} \begin{bmatrix} W_R X_R \\ Y_R Z_R \end{bmatrix} = \begin{bmatrix} K \quad 0 \\ 0 \quad -M \end{bmatrix} \quad (21) \]
\[ \begin{bmatrix} W_L X_L^T \\ Y_L Z_L^T \end{bmatrix} K_0 D_0 \begin{bmatrix} W_R X_R \\ Y_R Z_R \end{bmatrix} = \begin{bmatrix} 0 \quad K \\ K \quad D \end{bmatrix} \quad (22) \]

The matrices \( M, D, K \in \mathbb{R}^{n \times n} \) define the new system. Only constraint equations relating to two of the transformed \( A_0, B_0, C_0 \) need to be satisfied. Equations (20)–(22) contain \( 5n^2 \) independent constraints on the \( 8n^2 \) elements in the transformation matrices \( T_L, T_R \).

Solving the constraint equations (20)–(22) completely is very difficult, although it may be shown from equations (20)–(22) that all SPTs are of the form,

\[ T_L = \begin{bmatrix} F_L - \frac{1}{2} G_L D^T & -G_L M^T \\ G_L K^T & F_L + \frac{1}{2} G_L D^T \end{bmatrix} \quad (23) \]
\[ T_R = \begin{bmatrix} F_R - \frac{1}{2} G_R D & -G_R M \\ G_R K & F_R + \frac{1}{2} G_R D \end{bmatrix} \quad (24) \]

for some \( F_L, G_L, F_R, G_R \). Note that this form is necessary for the transformation to be an SPT, but not sufficient. Direct substitution of equations (23) and (24) into equations (20)–(22) produces the constraints on \( F_L, G_L, F_R, G_R \). The constraints for a symmetric system are given explicitly later.

If \( F_L \) and \( F_R \) are non-singular then

\[ T_L = \begin{bmatrix} F_L & 0 \\ 0 & F_L \end{bmatrix} \begin{bmatrix} I - \frac{1}{2} H_L D^T & -H_L M^T \\ H_L K^T & I + \frac{1}{2} H_L D^T \end{bmatrix} \quad (25) \]
\[ T_R = \begin{bmatrix} F_R & 0 \\ 0 & F_R \end{bmatrix} \begin{bmatrix} I - \frac{1}{2} H_R D & -H_R M \\ H_R K & I + \frac{1}{2} H_R D \end{bmatrix} \quad (26) \]

where \( H_L = F_L^{-1} G_L \) and \( H_R = F_R^{-1} G_R \).

The block diagonal matrices involving \( F_L, F_R \) in equations (25) and (26) represent conventional transformations, where only the physical degrees of freedom are transformed, and are a subset of all SPTs, where \( X_L = Y_L = 0 \) and \( W_L = Z_L \), and similarly for the right transformation. SPTs expand the set of transformations by allowing the transformations to mix displacements and velocities, known as unconventional transformations, and given by the last matrices in equations (25) and (26). Note also, that SPTs are state space transformations with constraints to maintain the second order nature of the system. Therefore SPTs...
are a subset of all state space transformations. Figure 1 shows this graphically. Transformations that are low rank modifications of the identity transformations may be defined for all types of transformation. These will be discussed in more detail later.

If the transformations are restricted to symmetry preserving transformation where \( T_L = T_R \), then \( H_L = H_R \) are skew-symmetric. This gives a simple way to generate an SPT, by choosing an arbitrary non-singular matrix \( F_L = F_R \) and an arbitrary non-singular, skew-symmetric matrix \( H_L = H_R \). If the transformations \( T_L = T_R \) are generated according to equations (25) and (26), then the result is an SPT.

In fact, more general definitions are possible for structure preserving transformations that allow rectangular and singular transformations on systems in which it is not necessary that any of the 3 coefficient matrices are non-singular. However these transformations are not applicable to isospectral flows, and for clarity further discussion of these is omitted here.

**DIAGONALISING SPTs**

Structure preserving transformations which diagonalise all three matrices \( M_0, D_0, K_0 \in \mathbb{R}^{n \times n} \) are of particular interest. Caughey [6] and Caughey & O’Kelly [2] investigated second order systems where all three matrices can be diagonalised simultaneously by transformations within the n-dimensional vector space. It is well known that in general there do not exist transformations \( \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \) which diagonalise all three matrices simultaneously. However, the situation is different if we consider structure preserving transformations.

Equations (20) and (21) gave the transformed state space matrices in terms of the transformed mass, damping and stiffness matrices \( M, D, K \in \mathbb{R}^{n \times n} \). If these matrices are diagonal then the transformation is called a diagonalising SPT. Although the ultimate aim is to dispense with complex eigenvalues and eigenvectors in the analysis of second order systems, Garvey et al. [1] gave a method to generate diagonalising SPTs from the complex eigenvectors of a system, provided that the generalised eigenvalue problem described by matrices \( A_0, B_0 \) is not defective (Lancaster [4] described such \( \lambda \)-matrices as *simple* in the case of non-repeated eigenvalues, or *semi-simple* otherwise).

Consider a self adjoint second order system, where the mass, damping and stiffness matrices are symmetric, and the left and right transformations are equal, \( T_L = T_R \). Further suppose that all of the modes are underdamped, so that none of the eigenvalues are real. The eigenvectors then occur in complex conjugate pairs, and the state space eigenvector matrix is

\[
\begin{bmatrix}
\Phi & \Phi \\
\Phi \Lambda & \Phi \Lambda
\end{bmatrix}
\]

where \( \Phi \) denotes the complex eigenvectors with positive imaginary part, \( \Lambda \) is a diagonal matrix of the corresponding eigenvalues, and the overbar denotes the complex conjugate. Garvey et al. [1] also considered the general case when some of the modes are overdamped. The SPT is then,

\[
T_L = T_R = \begin{bmatrix}
\Phi & \Phi \\
\Phi \Lambda & \Phi \Lambda
\end{bmatrix} J \Gamma \begin{bmatrix}
\Omega & 0 \\
0 & I
\end{bmatrix}.
\] (27)

In equation (27),

\[
J = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & -jI \\
I & jI
\end{bmatrix}.
\] (28)

where \( j = \sqrt{-1} \), produces a real matrix by combining the eigenvectors, using the fact they occur in complex conjugate pairs. If there are real eigenvectors then \( J \) also contains purely real or imaginary terms in the corresponding diagonal elements. The result of the application of the eigenvector matrix and \( J \) is a \( 2 \times 2 \) real block for each mode. Zeros are placed in the correct position using the matrix \( \Gamma \), where,

\[
\Gamma = \begin{bmatrix}
cosh(\gamma) & \sinh(\gamma) \\
\sinh(\gamma) & \cosh(\gamma)
\end{bmatrix}.
\] (29)

and \( \gamma \) is a real diagonal matrix, computed from the modal mass, damping and stiffness terms. Finally some scaling is performed using the last matrix of equation (27) to ensure that the modal mass (or stiffness) is the same in all of the transformed matrices. \( \Omega \) is the diagonal matrix of natural frequencies.

Thus, using SPTs any second order system may be decoupled into single degree of freedom systems, and there is no distinction between proportional and non-proportional damping. Indeed the example presented later shows that the extent of “non-proportionality” of a system can be changed dramatically using
isospectral flows. One of the primary motivations of SPTs is the development of algorithms which can progressively transform a system representation such that it ultimately approaches diagonal form (or more general canonical form) without ever leaving the space of second-order systems.

**ELEMENTARY SPTs**

Suppose that \( W_L, Z_L, W_R, Z_R \) are each unit-rank modifications of the identity matrix and \( X_L, Y_L, X_R, Y_R \) are unit-rank matrices. Under these restrictions, there must exist some eight \( n \)-vectors \( a_L, b_L, c_L, d_L, e_L, f_L, g_L, h_L \) defining the left transformation according to

\[
W_L = I + a_L b_L^T, \quad \quad Y_L = e_L f_L^T, \quad \quad X_L = c_L d_L^T, \quad \quad Z_L = I + g_L h_L^T.
\]

(30)

A further eight vectors define the right transformation. Substituting these expressions for transformations into the constraints obtained from equations (20)–(22) produces 5 conditions on the 16 vectors [7]. There are three classes of elementary SPTs (ESPTs) currently under investigation.

**Class 1 ESPTs**

Class 1 ESPTs are conventional transformations, given by

\[
W_L = I + a_L b_L^T, \quad \quad X_L = 0,
\]

\[
Y_L = 0, \quad \quad Z_L = I + a_L b_L^T.
\]

(31)

and

\[
W_R = I + a_R b_R^T, \quad \quad X_R = 0,
\]

\[
Y_R = 0, \quad \quad Z_R = I + a_R b_R^T.
\]

(32)

The changes in the mass, damping and stiffness matrices are all rank 2.

**Class 2 ESPTs**

Class 2 ESPTs are unconventional transformations, given by

\[
W_L = I + a_L b_L^T, \quad \quad X_L = a_L d_L^T,
\]

\[
Y_L = a_L f_L^T, \quad \quad Z_L = I + a_L h_L^T.
\]

(33)

and

\[
W_R = I + a_R b_R^T, \quad \quad X_R = a_R d_R^T,
\]

\[
Y_R = a_R f_R^T, \quad \quad Z_R = I + a_R h_R^T.
\]

(34)

A further six constraints need to be applied to ensure these are SPTs [7]. These constraints are,

\[
K_0^T a_L + x_K b_R + x_D f_R = 0
\]

(35)

\[
D_0^T a_L + x_K d_R + x_D (b_R + h_R) + x_M f_R = 0
\]

(36)

\[
M_0^T x_L + x_D d_R + x_M h_R = 0
\]

(37)

\[
K_0 a_R + x_K b_L + x_D f_L = 0
\]

(38)

\[
D_0 a_R + x_K d_L + x_D (b_L + h_L) + x_M f_L = 0
\]

(39)

\[
M_0 a_R + x_D d_L + x_M h_L = 0
\]

(40)

where

\[
x_K = a_L^T K_0 a_R
\]

\[
x_D = \frac{1}{2} a_L^T D_0 a_R
\]

\[
x_M = a_L^T M_0 a_R.
\]

Thus, four of the ten vectors in equations (33) and (34) may be chosen independently, and providing \( a_L \) and \( a_R \) are chosen, equations (35)–(40) form a linear set of equations for the other vectors. The changes in the mass and stiffness matrices are rank 2 and the change in the damping matrix is rank 4.

**Class 3 ESPTs**

Class 3 ESPTs are also unconventional transformations, given by

\[
W_L = I + (c_L - q_L) p_L^T, \quad \quad X_L = c_L d_L^T,
\]

\[
Y_L = c_L f_L^T, \quad \quad Z_L = I - (c_L + q_L) p_L^T,
\]

(42)

and

\[
W_R = I + (c_R - q_R) p_R^T, \quad \quad X_R = c_R d_R^T,
\]

\[
Y_R = c_R f_R^T, \quad \quad Z_R = I - (c_R + q_R) p_R^T.
\]

(43)

Again six further constraints need to be applied to ensure these are SPTs. These constraints are

\[
K_0 c_L - (t_K - x_K) p_R + x_K f_R = 0
\]

(44)

\[
D_0^T a_L - t_D p_R + x_K d_R + x_M f_R = 0
\]

(45)

\[
M_0^T c_L - (t_M + x_M) p_R + x_D d_R = 0
\]

(46)

\[
K_0 c_R - (s_K - x_K) p_L + x_D f_L = 0
\]

(47)

\[
D_0 c_R - s_D p_L + x_K d_L + x_M f_L = 0
\]

(48)

\[
M_0 c_R - (s_M + x_M) p_L + x_D d_L = 0
\]

(49)

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where

\[
\begin{align*}
x_K &= c_L^T K_0 c_R \\
x_D &= c_L^T D_0 c_R \\
x_M &= c_L^T M_0 c_R \\
T_K &= c_L^T K_0 q_R \\
T_D &= c_L^T D_0 q_R \\
T_M &= c_L^T M_0 q_R \\
\end{align*}
\]

The changes in the mass and stiffness matrices are rank 3 and the change in the damping matrix is rank 6.

**ISOSPECTRAL FLOWS**

Isospectral flows have been investigated extensively [8–10] to gain insight into the structure of numerical algorithms for solving the standard eigenvalue problem

\[ (A_0 - \lambda I_n)x = 0 \]

(51)

with \( A_0 \in \mathbb{C}^{n \times n} \), \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{C}^n \). Such algorithms are commonly based on a sequence of transformations \( T_i \in \mathbb{C}^{n \times n} \) applied to a given matrix \( A_0 \) to generate a sequence of isospectral matrices

\[ A_i := T_i^{-1}A_{i-1}T_i. \]

(52)

The eigenvalues of \( A_0 \) are identical to those of \( A_i \) for all \( i \). Moreover, where some eigenvalues have a multiplicity greater than 1, the dimensions of the corresponding eigenspaces are preserved. The form of the above equation suggests an extension to a differentiable one-parameter family of transformations \( T(t) \in \mathbb{C}^{n \times n} \forall t \in \mathbb{R} \), with \( T(0) = I_n \), which generates a corresponding family of isospectral matrices

\[ A(t) = T(t)^{-1}A_0T(t) \]

(53)

called an **isospectral flow**. Isospectral flows have been used directly in some cases as solution methods for eigenvalue problems (for example, [11–14]).

The derivative \( A(t) := dA(t)/dt \) can be expressed as

\[ \dot{A}(t) = A(t)N(t) - N(t)^{\top}(t)A(t) \]

(54)

where

\[ N(t) := T(t)^{-1}T(t) \]

(55)

defines a continuous one-parameter family of matrices. Conversely, given any continuous one-parameter family \( N(t) \) the solution of (54) will be of the form (53) for some differentiable \( T(t) \) satisfying

\[ T(t) = T(t)N(t). \]

(56)

The *Jacobi identity* shows that \( T(t) \) is non-singular for any integrable matrix function \( N(t) \), since \( T(0) = I \) is non-singular [15].

Isospectral flows have also been applied to the general eigenvalue problem arising from the matrix pencil (\( \lambda \)-matrix of degree 1)

\[ A_0\lambda + B_0. \]

(57)

Considering two one-parameter families of transformations \( T_L(t) \) and \( T_R(t) \in \mathbb{R}^{n \times n} \) we have

\[ A(t) = T_L^{-1}(t)A_0T_R(t) \]

(58)

\[ B(t) = T_L^{-1}(t)B_0T_R(t). \]

(59)

It is obvious that the generalised eigenvalues of \( (A(t), B(t)) \) remain unchanged provided that \( T_L(t) \) and \( T_R(t) \) are non-singular. It is also obvious that the dimensions of the eigenspaces of any repeated eigenvalues must be preserved. The associated differential equations are

\[ \dot{A}(t) = A(t)N_R(t) + N^\top_L(t)A(t) \]

(60)

\[ \dot{B}(t) = B(t)N_R(t) + N^\top_L(t)B(t) \]

(61)

with the definitions

\[ N_R(t) := T_R^{-1}(t)T_R(t) \]

(62)

\[ N_L(t) := T_L^{-1}(t)T_L(t). \]

(63)

The previous argument may be repeated. Given any two continuous one-parameter families, \( N_R(t) \) and \( N_L(t) \), the solutions \( A(t) \) and \( B(t) \) of (60) and (61) will be of the form (58) and (59) respectively (assuming \( T_L(t) \) and \( T_R(t) \) are non-singular). Thus \( (A(t), B(t)) \) must be isospectral with the original matrix pair \( (A(0), B(0)) \) for all \( t \). Gladwell [16] has applied this concept to investigate isospectral transformations constrained to preserve certain properties of undamped mechanical systems.
ISOSPECTRAL FLOWS FOR SECOND ORDER SYSTEMS

Given the real Lancaster pair \( A_0, B_0 \) of the regular \( \lambda \)-matrix \( \lambda^2 M_0 + \lambda D_0 + K_0 \) we consider the two one-parameter families \( T_L(t), T_R(t) \in \mathbb{R}^{2n \times 2n} \) of structure preserving transformations with \( T_L(0) = T_R(0) = I_{2n} \). Then the flows

\[
A(t) := T_L(t)A_0T_R(t) \quad (64) \\
B(t) := T_L(t)B_0T_R(t) \quad (65)
\]

represent an isospectral family of regular \( \lambda \)-matrices \( \lambda^2 M(t) + \lambda D(t) + K(t) \) if \( T_L(t) \) and \( T_R(t) \) satisfy

\[
T_R(t) = T_R(t) \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{bmatrix} \quad (66) \\
T_L(t) = T_L(t) \begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{21}(t) & Y_{22}(t) \end{bmatrix} \quad (67)
\]

for some one-parameter families \( X_{ik}(t), Y_{ik}(t) \in \mathbb{R}^{n \times n} \), for \( i, k = 1, 2 \) and if \( T_L(t) \) and \( T_R(t) \) preserve structure. Constraints apply to \( X_{ik}(t) \) and \( Y_{ik}(t) \) in order that \( T_L(t) \) and \( T_R(t) \) remain structure preserving. Equations (64)–(67) can be expanded to make clear these constraints. In the following equations all \( (n \times n) \) quantities are dependent on the parameter, \( t \), but the dependence is excluded for the sake of brevity. The derivative of \( A(t) \) is

\[
\begin{bmatrix} D & M \\ M & 0 \end{bmatrix} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix}^T \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}. \quad (68)
\]

Similar equations are obtained for \( B(t) \) and \( C(t) \). Expansion of these equations leads to \( 5n^2 \) linear constraints for the \( 8n^2 \) elements of \( X_{ik}, Y_{ik} \). It is trivial to demonstrate that these linear constraints are implicitly obeyed by defining

\[
X_{11} X_{12} = \begin{bmatrix} N_R 0 \\ 0 N_R \end{bmatrix} + \begin{bmatrix} L 0 \\ 0 L \end{bmatrix} - \frac{1}{2} D - M \\
X_{21} X_{22} = \begin{bmatrix} N_L 0 \\ 0 N_L \end{bmatrix} - \begin{bmatrix} L 0 \\ 0 L \end{bmatrix} - \frac{1}{2} D - M^T
\]

and

\[
Y_{11} Y_{12} = \begin{bmatrix} N_R 0 \\ 0 N_R \end{bmatrix} + \begin{bmatrix} L 0 \\ 0 L \end{bmatrix} - \frac{1}{2} D - M \\
Y_{21} Y_{22} = \begin{bmatrix} N_L 0 \\ 0 N_L \end{bmatrix} - \begin{bmatrix} L 0 \\ 0 L \end{bmatrix} - \frac{1}{2} D - M^T. \quad (70)
\]

where matrices \( N_R, L, N_L \in \mathbb{R}^{n \times n} \) can be chosen arbitrarily.

Theorem Every trajectory \( M(t), D(t), K(t) \) of regular \( \lambda \)-matrices of degree 2 is an isospectral flow if and only if

\[
\dot{K}(t) = K(t)N_R(t) + N_R^T(t)K(t) + \frac{1}{2} [D(t) L(t) K(t) - K(t) L(t) D(t)] \quad (71)
\]

\[
\dot{M}(t) = M(t)N_R(t) + N_R^T(t)M(t) + \frac{1}{2} [M(t) L(t) D(t) - D(t) L(t) M(t)] \quad (72)
\]

\[
D(t) = D(t)N_R(t) + N_R^T(t)D(t)
\]

where \( N_R(t), L(t), N_L(t) \in \mathbb{R}^{n \times n} \) are three continuous one-parameter families of matrices.

Proof: Substituting (69) and (70) into (68) and the equivalent equations for the derivative of \( B \) and \( C \) clearly demonstrates that equations (71)–(73) are satisfied, and thus that (71)–(73) constitute an isospectral flow.

The remaining question is whether there are any other isospectral flows not accounted for by (71)–(73). It is straightforward to start with the constraints implied by (68) and the equivalent equation for the derivative of \( B \) and to solve for the \( X_{ik} \) and \( Y_{ik} \) to obtain (69) and (70). However the form of the matrices given by (69) and (70) are unique only if \( M(t) \) is non-singular for all \( t \).

Garvey et al. [5] showed that \( M(0) \) is invertible if and only if \( A(0) \) is invertible. Invoking the Jacobi Identity [15], in view of (69) and (70), delivers a guarantee that matrices \( T_L(t) \) and \( T_R(t) \) are non-singular. Hence \( A(t) \) is non-singular for all \( t \) and it follows that \( M(t) \) is also non-singular for all \( t \).

In equations (71)–(73) the conventional flow is given by the first two terms on the right side of each equation, whereas the unconventional flow is given by the last two terms.

We conclude this section with a remark on symmetries. Any symmetries of \( M, D \) or \( K \) must also apply for their derivatives. From (71) to (73) we may deduce:

1. If \( M, D, K \) all symmetric, then symmetry is preserved through setting \( N_L(t) = N_R(t) \) and \( L(t) = -L^T(t) \).
2. If \( M, K \) symmetric and \( D \) skew symmetric, this property is preserved through setting \( N_L(t) = N_R(t) \) and \( L(t) = L^T(t) \).
3. If \( M, D, K \) all skew-symmetric, this property is preserved through setting \( N_L(t) = N_R(t) \) and \( L(t) = -L^T(t) \).
4. If \( M, K \) skew-symmetric and \( D \) symmetric, this property is preserved through setting \( N_L(t) = N_R(t) \) and \( L(t) = L^T(t) \).

EXAMPLE

Only one example is given here, and the matrices \( N_L, L, N_R \in \mathbb{R}^{n \times n} \) parameterising the flow are independent of the parameter, \( t \), and the system is self-adjoint throughout the flow. An arbitrary isospectral flow is carried out on a system that is originally diagonal (and hence it is implicitly classically damped). The flow is parameterised by three constant matrices \( N_L, L, N_R \in \mathbb{R}^{n \times n} \) with \( L \) being non-zero. As a result of this, the extent of non-proportionality of the system [17] changes. Perhaps the most interesting aspect of this example is that if the flow is operated...
backwards from the final system \( t = 1 \), the original diagonal system is restored. This is the first instance of an isospectral flow being used to simultaneously diagonalise the three coefficient matrices of a \( \lambda \)-matrix of degree 2. The original system in this case is deliberately chosen to have \( K(0) \) singular and the numerical integrations bear out the expectation that \( K(t) \) is singular for all \( t \).

The original system is given by

\[
K(0) = diag[25 \hspace{0.2cm} 9 \hspace{0.2cm} 0] \tag{74}
\]

\[
D(0) = diag[10 \hspace{0.2cm} 20 \hspace{0.2cm} 30] \tag{75}
\]

\[
M(0) = diag[1 \hspace{0.2cm} 4 \hspace{0.2cm} 16] \tag{76}
\]

The matrices \( N_L, L, N_R \) parameterising the flow are given by

\[
N_L = \begin{bmatrix} 0 & 5 & 6 \\ -5 & 0 & -5 \\ -6 & 5 & 0 \end{bmatrix} = N_R \tag{77}
\]

and

\[
L = \begin{bmatrix} 0 & 0.1 & 0 \\ -0.1 & 0 & -0.1 \\ 0 & 0.1 & 0 \end{bmatrix} \tag{78}
\]

Figures 2 to 4 show the trajectories of entry \( (1,1) \) of each of \( K(t), D(t), M(t) \) respectively. At \( t \approx 0.236 \) entry \( (1,1) \) of \( K(t) \) comes close to zero but it cannot cross the zero (its lowest value in Figure 2 is 0.013). The rank of \( K(t) \) is 2 throughout the integration. Figure 5 shows a plot of non-proportionality as defined by Tong et al. [18] varying with respect to \( t \). Its initial value is zero.

Matrices \( K(1), D(1), M(1) \) are found to be

\[
K(1) = \begin{bmatrix} 2.581876 & -12.148104 & -5.094329 \\ -12.148104 & 216.569403 & 42.993048 \\ -5.094329 & 42.993048 & 12.321870 \end{bmatrix} \tag{79}
\]

\[
D(1) = \begin{bmatrix} 8.540700 &44.416948 & -1.396424 \\ 44.416948 & 211.412494 & 92.154337 \\ -1.396424 & 92.154337 & 37.272603 \end{bmatrix} \tag{80}
\]

\[
M(1) = \begin{bmatrix} 12.381402 & 23.087974 & 13.949189 \\ 23.087974 & 51.290516 & 35.010173 \\ 13.949189 & 35.010173 & 26.173014 \end{bmatrix} \tag{81}
\]

An interesting aspect to the above matrices is that \( D(1) \) has one negative eigenvalue and yet the system is stable.

Performing only a slight modification to the initial conditions in this example produces an illustration of the complete generality of the isospectral flow equations. Setting \( M(0) = diag[0 \hspace{0.2cm} 4 \hspace{0.2cm} 16] \) and \( D(0) = diag[1 \hspace{0.2cm} 0 \hspace{0.2cm} 3] \) results in a \( \lambda \)-matrix of grade 2 in which all of the coefficient matrices are singular. The flow equations can nevertheless be integrated without difficulty and a family of isospectral systems is obtained in which \( M(t) \) and \( K(t) \) are both singular for all \( t \). Conventional definitions of isospectrality cannot be applied to such systems.

**CONCLUSIONS**

This paper discusses general coordinate transformations for linear second order systems. It notes that the full set of possible coordinate transformations for linear second order systems includes as a major subset the set of structure-preserving transformations. This set of transformations includes as a subset the set of all first-order coordinate transformations which coincide with the view of coordinate transformations established in the structural dynamics community. Within the set of structure-preserving coordinate transformations for (almost) any given system, there is a transformation involving only real numbers that transforms the original system into a new form in which the system matrices are real and diagonal. The only exceptions are defective systems. A route to the determination of this particular diagonalising transformation has been given beginning with solution of the well-known eigenvalue problem in complex numbers.

This paper has also defined the concept of an elementary structure-preserving transformation for a general linear second order system. It has given three distinct classes of transformations, and one of these contains all of the conventional coordinate transformation matrices that are unit rank modifications of the identity transformation. Transformations within the other classes involve combinations of displacements and velocities from the original system coordinates in the displacement coordinates of the transformed system. In numerical processes, the ability to transform from a general (self-adjoint) system to a tridiagonal form and subsequently to a diagonal form may completely supplant the existing numerical methods for computing eigenfrequencies and modes. Such a transformation is expected to follow a close parallel to a process already developed for simultaneously tridiagonalising the two matrices of an undamped system [19]. Isospectral flows for general linear second order systems have also been introduced, and these may provide a method to computing these transformations. Tridiagonal system form may be especially useful for dynamic substructuring applications. The possibility of performing structure-preserving model-reducing transformations constructed from elementary structure-preserving transformations in the course of a finite-element cal-
calculation (model assembly) also appears very strong.

The implications of the paper are many. The use of structure-preserving transformations for linear second order systems may ultimately lead to improved computational performance in the determination of system characteristic behaviour - possibly through constructing the diagonalising transformation as the product of a large number of elementary structure-preserving transformations. More importantly, though, it shows some prospects for substantially improved clarity in the study of generally-damped systems. Applications include,

- Deflating a second order system knowing a pair of complex modes [20]
- Modal correlation measures which can detect if errors are confined to one of the system matrices [21]
- Selective excitation of system modes using a limited number of shakers [22]
- Model reduction for non-proportional systems [23]
- Analysis of systems which have a (near) singular mass matrix and even non-square systems

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REFERENCES

[1] SD Garvey, MI Friswell, and U Prells. Coordinate transformations for second-order systems: Part i general transfor-


