The Dynamics of Rotating Machines with Cracks

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Keywords rotor, crack

Abstract There are a number of approaches reported in the literature for modelling cracks in shafts. Although two and three dimensional finite element models may be used, the most popular approaches are based on beam models. Of particular concern is the modelling of breathing cracks, which open and close as the rotor spins due to its self-weight. One simple model of a breathing crack assumes the shaft stiffness varies as a cosine function as the crack opens and closes. In large rotating machines the deflection caused by out-of-balance forces is small compared with that due to gravity and also the cracks are, at least initially, small. This paper will consider the influence of these two assumptions on the response of a cracked Jeffcott rotor. The effect of an asymmetric shaft is also considered.

1. Introduction

The idea that changes in a rotor’s dynamic behaviour could be used for general fault detection and monitoring was first proposed in the 1970s. Of all machine faults, probably cracks in the rotor pose the greatest danger and research in crack detection has been ongoing for the past 30 years.

If the vibration due to any out-of-balance forces acting on a rotor is greater than the static deflection of the rotor due to gravity, then the crack will remain either opened or closed depending on the size and location of the unbalance masses. In the case of the permanently opened crack, the rotor is then asymmetric and this condition can lead to stability problems. Of more importance in large machines is the situation where the vibration due any out-of-balance forces acting on a rotor is less than the deflection of the rotor due to gravity. In this case the crack will open and close (or breathe) due to the turning of the rotor. The problem was initially studied by Gasch [1] who modelled the breathing crack by a “hinge”. In this model the crack is opened for one half and closed for the other half a revolution of the rotor. The transition from opened to closed (and visa-versa) occurs abruptly as the rotor turns. Mayes and Davies [2,3] developed a similar model except that the transition from fully opened to fully closed is described by a cosine function (see section 2). Relating crack size to shaft stiffness is not easy and Papadopoulos and Dimarogonas [4] and Jun et al. [5] have proposed crack models based on fracture mechanics. Papadopoulos and Dimarogonas [4] provide a full 6 x 6 flexibility matrix for a transverse surface crack. The best crack model is still an unresolved issue and Keiner and Gadala [6] have compared the natural frequencies and orbits of a simple cracked rotor using a beam model and two possible FE models of the crack.

Gasch [7] has carried out survey of the stability of a simple rotor with a crack and also considered the response of the rotor due to unbalance. Gasch also shows that as long as the resulting vibrations remain small the essentially non-linear equations of motion become linear with periodically time varying coefficients. Pu [8] also considers a self-weight dominated rotor and solves the resulting equations using the harmonic balance method.
2 The Mayes model for an opening and closing crack

This model is defined, in rotating coordinates, by

\[ k_\eta(\theta) = k_{M\eta} + k_{D\eta}C_1 \]  

(1a)

and

\[ k_\zeta(\theta) = k_{M\zeta} + k_{D\zeta}C_1 \]  

(1b)

where \( k_{M\zeta} = \frac{1}{2}(k_0 + k_{\zeta}) \), \( k_{D\zeta} = \frac{1}{2}(k_0 - k_{\zeta}) \), etc., and \( C_1 = \cos(\theta) \). When \( \cos(\theta) = 1 \) the crack is closed and \( k_\eta(\theta) = k_\zeta(\theta) = k_0 \). When \( \cos(\theta) = -1 \) the crack is fully open and \( k_\eta(\theta) = k_\zeta \), \( k_\zeta(\theta) = k_{\zeta} \).

The stiffnesses in fixed coordinates can be determined by transforming from rotating coordinates so that

\[ K_F = T^T K_R T \]  

(2)

where

\[ T = \begin{bmatrix} C_1 & S_1 \\ -S_1 & C_1 \end{bmatrix} \]  

(3)

In (3), \( S_1 = \sin(\theta) \). Carrying out the matrix multiplications of (2) and expanding the trigonometric expressions in multiple angles gives

\[ (K_F)_{12} = \frac{1}{2}(k_{M\zeta} - k_{M\eta})S_2 + \frac{1}{4}(k_{D\zeta} - k_{D\eta})(S_1 + S_3) \]  

or

\[ (K_F)_{12} = \frac{1}{4}(k_{\zeta} - k_\eta)(S_2 - \frac{1}{2}(S_1 + S_3)) \]  

(4)

where \( S_3 = \sin(3\theta) \), etc. Similarly

\[ (K_F)_{11} = k_0 - \frac{1}{4}(k_{D\zeta} + k_{D\eta}) - \frac{1}{2}(k_{D\zeta} - k_{D\eta})C_2 + \frac{1}{4}\left(3k_{D\zeta} + k_{D\eta}\right)C_1 + \left(3k_{D\zeta} - k_{D\eta}\right)C_3 \]  

(5)

and

\[ (K_F)_{22} = k_0 - \frac{1}{2}(k_{D\eta} + k_{D\zeta}) - \frac{1}{2}(k_{D\eta} - k_{D\zeta})C_2 + \frac{1}{4}\left(3k_{D\eta} + k_{D\zeta}\right)C_1 + \left(k_{D\eta} - k_{D\zeta}\right)C_3 \]  

(6)

Now, if \( \theta = \Omega t \), where \( \Omega \) is the rotor angular velocity, then in fixed coordinates the Mayes model generates a constant term plus 1X, 2X and 3X rotor angular velocity components in the diagonal stiffness terms and 1X, 2X and 3X rotor angular velocity components in the off-diagonal stiffness terms.

3. Approximating the Equations of motion

The analysis may be performed in fixed or rotating coordinates. If the bearings and foundations are axi-symmetric then the stator dynamic stiffness will appear constant in the rotating frame, and there is some benefit in analysing the machine in rotating coordinates. Typically foundations will be
stiffer vertically than horizontally, and in this case the advantage in using rotating coordinates is significantly reduced. Thus in this paper the machine is analysed in fixed coordinates.

In fixed coordinates we have \( K(0,t) = K_0 - K_D(0,t) \) where \( K_0 \) is the diagonal stiffness matrix for the undamaged beam and \( K_D(0,t) \) is the stiffness change due to damage. \( \theta \) is the angle between the crack axis and the rotor response at the crack location and determines the extent to which the crack is open. Let the deflection the system be \( q = q_{st} + q_u \) where \( q_{st} \) is the static deflection and \( q_u \) is the deflection due to whirl. Thus, \( \ddot{q} = \ddot{q}_u \) and \( \dot{q} = \dot{q}_u \), and the equations of motion in fixed coordinates is

\[
M \ddot{q}_u + (D + G) \dot{q}_u + (K_0 - K_D(0,t))(q_{st} + q_u) = Q_u + W
\]

(7)

where \( Q_u \) and \( W \) are the out of balance forces and gravitational force respectively. Damping and gyroscopic effects have been included as a symmetric positive semi-definite matrix \( D \) and a skew-symmetric matrix \( G \), although they have little direct bearing on the analysis. If there is axisymmetric damping in the rotor then there will also be a skew-symmetric contribution to the undamaged stiffness matrix, \( K_0 \). We refer to (7) as the “full equations”.

There are two approximations that are commonly used in the analysis of cracked rotors, namely weight dominance and neglecting the parametric excitation terms. These will be dealt with in turn.

**Weight dominance.** One common assumption is that the static deflection is much greater than the response due to the unbalance or rotating asymmetry, that is \( |q_{st}| \gg |q_u| \). For example, for a large turbine rotor the static deflection might be of the order of 1 mm whereas at running speed the amplitude of vibration is typically 50 \( \mu \)m. Even at a critical speed the allowable level of vibration will only be 250 \( \mu \)m. In this situation that the crack opening and closing is dependent only on the static deflection and thus that \( \theta = \Omega t + \theta_0 \), where \( \Omega \) is the rotor speed and \( \theta_0 \) is the initial angle. Thus this case \( K_D(0,t) = K_D(t) \) is a period function of time and the full non-linear equation (7) becomes a linear parametrically excited equation. We refer to this as the “weight dominance assumption”.

**Neglecting the parametric excitation terms.** Generally cracks are small, and so not only does \( |q_{st}| \gg |q_u| \), also \( |K_0| \gg |K_D(0)| \). This means that the term \( K_D(0,t)q_u \) is second order and may be ignored. Since, from the static solution, \( K_0 q_{st} = W \), equation (7) becomes,

\[
M \ddot{q}_u + (D + G) \dot{q}_u + K_0 q_u = Q_u + K_D(0,t)q_{st}
\]

(8)

which is a forced linear system, if weight dominance is also assumed. In this case we refer to (8) as “weight dominance ignoring parametric excitation”. If weight dominance is not assumed, then the crack is opening and closing, and hence \( K_D(0,t) \) depends on the response in a non-linear way. For a stability analysis the parametric terms are very important and must be included [7], although for a parametrically excited system the stability is not affected by the steady state solution, which therefore does not have to be calculated.
4. A Jeffcott rotor example

A Jeffcott rotor modelled using two degrees of freedom will be used to demonstrate the effect of the different assumptions in the equations of motion of the cracked rotor. The rotor is 700 mm long, with a diameter of 15 mm, and has a 1 kg disc at the centre. The crack is assumed to be located close to the disc and only responses that are symmetrical about the shaft centre are considered. The natural frequency of the rotor is approximately 42 Hz the damping ratio is taken as 1%. Gyroscopic effects are neglected. An unbalance force arising from a 10 g mass at 100 mm radius is used to excite the rotor, and at $t = 0$ the crack is closed and the unbalance force is in the $x$ direction.

To consider the effect of weight dominance consider a crack of 40% of the diameter, modelled using the Mayes approach with stiffness reduction for the fully open crack of 9% and 4% in directions perpendicular and tangential to the crack front. These reductions in direct stiffness due to the open crack are taken directly from the results of the model of Jun et al. [5]. A rotor speed of 1240 rev/min is simulated, which is just below half of the machine’s natural frequency. The static deflection of the undamaged rotor is 0.14 mm. The time simulations are run until the transient motion becomes negligible and the steady state solution remains. Figure 1 compares the orbit of the full solution of the equations of motion (solid) and the weight dominated solution (dotted) when there is no unbalance force. Clearly the assumption of weight dominance is a good one. When the unbalance increases to $10^{-4} \text{kg m}$ the assumption of weight dominance is not so good, and there is clearly some difference in the phase when the crack opens and closes (Figure 2). By the time the unbalance has increase to $10^{-3} \text{kg m}$ there is a qualitative difference in the weight dominated solution, which still shows a significant 2X response, and the full solution which shows predominantly a 1X response (Figure 3). Note that with unbalance of $10^{-4} \text{kg m}$ the vertical displacement never becomes positive, that is the response from the unbalance is always a smaller magnitude than the static deflection due to gravity. This is not so for an unbalance of $10^{-3} \text{kg m}$.

Figure 1. Comparison of the rotor orbit with no unbalance. Full equations (solid), weight dominance assumption (dotted). All dimensions in mm.

Figure 2. Comparison of the rotor orbit with an unbalance of $10^{-4} \text{kg m}$. Full equations (solid), weight dominance assumption (dotted). All dimensions in mm.
Figure 3. Comparison of the rotor orbit with an unbalance of $10^{-3}$ kg m. Full equations (solid), weight dominance assumption (dotted). All dimensions in mm.

Figure 4. Comparison of the rotor orbit with no unbalance for a large crack. Full equations (solid), weight dominance ignoring parametric excitation (dotted). All dimensions in mm.

Figure 4 compares the response of the full equations with those where the parametric excitation has been ignored. Clearly there is a large difference, although it could be argued that a crack of 40% of the diameter of the shaft is every large. Figure 5 compares the responses when the crack is much smaller, simulated making the stiffness reduction one tenth of that due to a 40% crack. Clearly the response are now very close.

5. An Asymmetric Shaft

Consider a rotor with asymmetric stiffness properties $k_H$ and $k_L$ in the orthogonal directions ($\sigma$, $\tau$) and assume that a crack develops with a crack face at an angle $\alpha$ to the direction $\sigma$. For simplicity a rotor with a circular section is assumed. Let the rotor stiffnesses in the ($\sigma$, $\tau$) rotating coordinates be $k_H$ and $k_L$. In the ($\xi$, $\eta$) rotating coordinates the stiffness matrix becomes

$$
\begin{bmatrix}
H_L & L_S & C \\
H_S & L_H & C \\
C & C & C
\end{bmatrix}
$$

Consider for simplicity the Mayes crack model. Then the stiffnesses in the ($\xi$, $\eta$) directions when the crack is open are

$$
\begin{bmatrix}
k_{\xi} - \Delta k_{\xi}(t) & 0 \\
0 & k_{\eta} - \Delta k_{\eta}(t)
\end{bmatrix}
$$

where $\Delta k_{\eta}$ (for example) is the reduction in stiffness due to the open crack in the $\eta$ direction. In this case we have

$$
\begin{bmatrix}
k_H C^2 + k_L S^2 - \Delta k_{\xi} & (k_H - k_L) S_{\alpha} C_{\alpha} \\
(k_H - k_L) S_{\alpha} C_{\alpha} & k_H S^2 + k_L C^2 - \Delta k_{\eta}(t)
\end{bmatrix}
$$
Let $k_{0\xi} = k_H C_\alpha^2 + k_L S_\alpha^2$, $k_{0\eta} = k_H S_\alpha^2 + k_L C_\alpha^2$ and $k_{0\xi\eta} = (k_H - k_L) S_\alpha C_\alpha$.

Assume a Mayes model, and let

$$k_{M\xi} = \frac{(k_{0\xi} + k_{\xi})}{2}, \quad k_{D\xi} = \frac{(k_{0\xi} - k_{\xi})}{2}$$

$$k_{M\eta} = \frac{(k_{0\eta} + k_{\eta})}{2}, \quad k_{D\eta} = \frac{(k_{0\eta} - k_{\eta})}{2}$$

Thus equation (11) becomes

$$\mathbf{K}_{Re} = \begin{bmatrix} k_{M\xi} + k_{D\xi} C_1 & k_{0\xi\eta} \\ k_{0\xi\eta} & k_{M\eta} + k_{D\eta} C_1 \end{bmatrix} \quad (12)$$

Now in fixed coordinates the stiffness matrix is given by

$$\mathbf{K}_{Fc} = \mathbf{T}^T \mathbf{K}_{Re} \mathbf{T} \quad (13)$$

This leads to the following expressions

$$(k_F)_{12} = \frac{1}{4} \left( k_{D\xi} - k_{D\eta} \right) \left( S_1 + S_3 \right) + \frac{1}{2} \left( k_{M\xi} - k_{M\eta} \right) S_2 + k_{0\xi\eta} C_2$$

$$(k_F)_{11} = \frac{1}{2} \left( k_{M\xi} + k_{M\eta} \right) + \frac{1}{4} \left( 3 k_{D\xi} + k_{D\eta} \right) C_1 + \frac{1}{2} \left( k_{M\xi} - k_{M\eta} \right) C_2 + \frac{1}{4} \left( k_{D\xi} - k_{D\eta} \right) C_3 - k_{0\xi\eta} S_2$$

and

$$(k_F)_{22} = \frac{1}{2} \left( k_{M\xi} + k_{M\eta} \right) + \frac{1}{4} \left( k_{D\xi} + 3 k_{D\eta} \right) C_1 - \frac{1}{2} \left( k_{M\xi} - k_{M\eta} \right) C_2 - \frac{1}{4} \left( k_{D\xi} - k_{D\eta} \right) C_3 + k_{0\xi\eta} S_2$$

Note that this isn’t quite as simple as the symmetric rotor because of the additional $k_{0\xi\eta}$ term and also because $k_{0\xi} \neq k_{0\eta}$. In the symmetric rotor $k_{0\xi\eta} = 0$ and $k_{0\xi} = k_{0\eta} = k_0$.

Figure 6 shows the response of the Jeffcott rotor described in the previous section for an unbalance of $10^{-4}$ kg m, as the solid line. This is the same response as given in Fig. 2. Also shown (dashed) is the response of the uncracked shaft, but with a 3% stiffness asymmetry in the shaft and $\alpha=0$. Although the responses are quantitatively different, both have a significant 2X component, as expected, and they have similar magnitudes. However when the response of the cracked asymmetric shaft is computed (dotted) the response magnitude is smaller, and the 2X component of the response is significantly reduced. If the cracked shaft were monitored, then as the crack propagated there would be a reduction in the 2X response, before the crack grew sufficiently to dominate the response.
Conclusions

In this study the steady state orbits of a cracked Jeffcott rotor at a speed just below half the rotor natural frequency are simulated and different crack model assumptions are compared. The static deflection of the rotor is 0.14 mm. With no unbalance the weight dominance assumption is satisfactory and the orbit shows predominantly a 2X response. With an unbalance of $4 \times 10^{-4}$ kg m the 2X response is less predominant and weight dominance assumption is less good, but still acceptable. With an unbalance of $3 \times 10^{-3}$ kg m the weight dominance assumption is no longer valid and the whirl orbit is larger than the static deflection. The orbit derived from the solution of the full equations shows predominantly a 1X response whereas the weight dominance assumption clearly contains some 2X response. Ignoring the parametric excitation terms is acceptable only if the crack is small so that the amplitude of the whirl orbit is small compared with the static deflection. If this condition is not met the parametric excitation terms cannot be ignored.

With asymmetric shafts there is the possibility that the 2X response can reduce as the crack grows. This is counter-intuitive, and is critically dependent on the phase of the crack, the shaft asymmetry and the unbalance.

References


