The Propagation of Parameter Uncertainty through Structural Dynamics Models

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ABSTRACT

During structural dynamic analysis uncertainty is often present in structural parameters such as the material properties and dimensions. A review of uncertainty propagation methods is made. The methods are compared in terms of accuracy, efficiency and applicability, and the Monte Carlo method is taken as the reference method. The test cases utilised are a cantilever beam and a plate.

1 INTRODUCTION

As computers become more powerful there has been an increasing emphasis on the modelling of structures within the design stage in order to reduce the costs of the development program. However, it is increasingly apparent that the deterministic nature of the modelling methods employed for low frequency structural dynamics modelling have serious drawbacks. The variance in the noise and vibration response of structures, such as vehicles, is still a major concern and large resources are expended to identify and remove sources of variability.

Many engineers believe that any structure may be modelled to arbitrary accuracy merely by increasing the finite element mesh density. But this is not so. These refined meshes are able to model the geometry of the structure more accurately, but uncertain parameters mean that modelling errors will never be resolved by this approach.

The deterministic finite element method is well established, and is successfully applied to a wide class of engineering problems. Attempts to develop a stochastic finite element method are not new, but no method so far has reached the same level of universal applicability. Despite being quite often considered a method of last resort, the Monte Carlo method is still one of the most successful, mostly due its generality and ease of implementation. Unfortunately it is too time consuming for most applications. To overcome this limitation researchers have tried to develop other methods which can yield faster results. Two of the most common methods are analysed: the perturbation method, and the fuzzy finite element method.

2 THE MONTE CARLO SIMULATION METHOD

The Monte Carlo method is applied in many different fields of computational science to problems with and without probabilistic content. It provides approximate solutions by performing statistical sampling experiments.

One of its major advantages is that accurate solutions can be obtained for problems whose deterministic solution is known. Since it is completely general this method is frequently used to calibrate and validate other methods. The main disadvantage is that it is time consuming, but among all numerical methods that rely on $n$-point evaluations in a $d$-dimensional space to produce an approximate solution, the Monte Carlo method has an absolute error estimation that decreases as $n^{-1/2}$, while in the absence of an exploitable special structure all others methods have an error estimation that decreases as $n^{-1/d}$ at best. Since each realisation is independent of the others Monte Carlo simulations can be easily parallelised.
3 THE PERTURBATION METHOD

The perturbation method, which is equivalent to a lower-order Taylor expansion, has been widely used for its tractability and computational time-saving [1]. It expresses the structural matrices and response in terms of a lower-order polynomial function with respect to the parameters centred at the mean values, i.e., makes an approximation of the response surface. Introducing the following notation

\[ \Phi^0 = \Phi(\mu_1, \mu_2, \ldots, \mu_n) \] (1)

\[ \Phi^I = \frac{\partial \Phi}{\partial x_i}(\mu_1, \mu_2, \ldots, \mu_n) \] (2)

\[ \Phi^{II}_{ij} = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(\mu_1, \mu_2, \ldots, \mu_n) \] (3)

for any variable \( \Phi \) which depends of the parameters \( x_1, x_2, \ldots, x_n \) then the stiffness and mass matrices are expanded as

\[ K = K^0 + \sum_{i=1}^{n} K^{Ii}_i \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} K^{II}_{ij} \epsilon_i \epsilon_j + \ldots \] (4)

\[ M = M^0 + \sum_{i=1}^{n} M^{Ii}_i \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M^{II}_{ij} \epsilon_i \epsilon_j + \ldots \] (5)

where \( \epsilon_i = x_i - \mu_i \). In the same manner, for the eigenvalue problem

\[ (K - \lambda M)\phi = 0 \] (6)

the resulting eigenvalues \( \lambda \) and eigenvectors \( \phi \) are expanded as

\[ \lambda = \lambda^0 + \sum_{i=1}^{n} \lambda^{Ii}_i \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^{II}_{ij} \epsilon_i \epsilon_j + \ldots \] (7)

\[ \phi = \phi^0 + \sum_{i=1}^{n} \phi^{Ii}_i \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi^{II}_{ij} \epsilon_i \epsilon_j + \ldots \] (8)

The mean and variances of the eigenvalues are

\[ \text{E}[\lambda] = \lambda^0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^{II}_{ij} \text{Cov}[x_i, x_j] + \ldots \] (9)

\[ \text{Var}[\lambda] = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^{Ii}_i \lambda^{Ij}_j \text{Cov}[x_i, x_j] + \ldots \] (10)

It can be shown that \( \lambda^{Ii} \) is given by [2][3][4]

\[ \lambda^{Ii}_i = \frac{\phi^{0T} (K^0 - \lambda^0 M^0) \phi^0}{\phi^{0T} M^0 \phi^0} \] (11)

where \( \lambda^0 \) and \( \phi^0 \) are obtained by the mean eigenvalue problem

\[ (K^0 - \lambda^0 M^0)\phi^0 = 0 \] (12)

The natural frequencies can be expanded as

\[ \omega = \sqrt{\lambda} \]

\[ = \mu^2 + \frac{1}{2} \mu^2 (\lambda - \mu) \]

\[ - \frac{1}{8} \mu^2 (\lambda - \mu)^2 + \ldots \] (13)

Therefore considering the up to the first order term, their mean and standard deviation can be obtained from equations (9) and (10) as

\[ \mu_\omega = \sqrt{\mu_\lambda} \] (14)

\[ \sigma_\omega = \frac{1}{2} \sqrt{\mu_\lambda} \sigma_\lambda \] (15)

It is required that the random variables involved in the analysis do not deviate much from their expected values. If the coefficient of variation is not very small the solution cannot ever be improved by using a finer mesh [5]. Of greater importance than the magnitude of variability of the original random variables is how appropriate the response surface is. The response quantity should be chosen in such a way that these quantities are not highly nonlinear in respect to the random variables, e.g., a frequency response function instead of a response in time domain [1].

If the number of random variables is large, such as in problems involving random fields, or if a high-order expansion is used, then the calculation effort becomes prohibitive.

4 FUZZY FINITE ELEMENT METHOD

Fuzzy sets model uncertainties due to vague definition rather than by chance. It uses membership functions as a replacement for probability density functions. It is often possible to construct these membership functions if expert knowledge exists, but they can also be obtained from the probability density functions [6][7].

For numerical computation, a fuzzy number is approximated by a set of closed intervals that correspond to specific \( \alpha \)-cuts of the membership function. The value of the membership function \( \alpha \) indicates a degree of confidence about the numerical value of the variable \( \alpha \) leading to a parameterised interval description (see figure 1). Fuzzy arithmetic operations can be carried out by using interval operations at each of the \( \alpha \)-levels independently, so the

![Figure 1: Interval representation of a fuzzy number \( \alpha \)](image-url)
membership functions have a simpler and computational more efficient algebra than the probability density functions. Membership functions must also be convex – they cannot represent bi-modal variables. In this work we approximate the Gaussian random variables by sampling the normalised PDF and equalising the underlying area, as seen in figure 2.

Figure 2: Approximation of a Gaussian distribution by a fuzzy number

Due to the lack of certain properties of fuzzy numbers (for example fuzzy multiplication is, in general, not distributive) almost all classical numerical techniques can’t be directly extended to fuzzy equations. A major computational difficulty with fuzzy arithmetic is the problem of overestimating the interval widths of the response quantities when the problem involves multiple occurrences of the same variable [8].

The most common fuzzy finite element approach is to replace the crisp stiffness and mass matrices by fuzzy-valued matrices, and then solve the interval eigenvalue problem at each \( \alpha \)-level, but this was shown to be very ineffective. It is assumed that the variation of eigenvalues \( \lambda \) with respect to the parameters \( x \) is monotonic within the parameter interval \( [x_{\alpha}, x] \) for each \( \alpha \)-level. Therefore the eigenvalue lower bound \( \lambda_l \) and upper bound \( \lambda_u \) for each \( \alpha \)-level is obtained by choosing the combination of the parameters upper and lower bounds according with the sign of the eigenvalue partial derivative \( \lambda_I \). The procedure for the one \( \alpha \)-level of a single parameter case is illustrated on figure 3.

5 OTHERS METHODS

This paper is not intended to give an extensive review of the available methods for uncertainty propagation in structural dynamics. There are several variants of these methods outlined which try to address some of their limitations. The Karhunen-Loève decomposition is frequently used together with these methods to make a reduction of the number of parameters [11]. There are variations of the Monte Carlo method which can achieve better convergence [9]. Elishakoff [10] proposed an improved first-order perturbation approach which can yield more accurate results.

6 RANDOM FIELDS

Uncertain properties like Young’s modulus, mass density or plate thickness vary in space, so they cannot be represented by a single variable. Instead, they are described as a random field. A random field \( X(t) \) is a collection of random variables at points with coordinates \( t = (t_1, \ldots, t_n) \) in an \( n \)-dimensional parameter space. In most engineering applications the random fields can be considered homogeneous, i.e., all the joint probability distribution functions remain the same when the set of locations is translated in the parameter space. We will also consider that the random fields in question are isotropic, i.e., all the joint probability distributions functions remain the same when the set of locations is rotated in parameter space. These two properties mean that the correlation between the random field value at two locations depends only on the distance between these two points.

There are several random field models with the above properties. The one used will be the first order auto-regressive model AR(1). The coefficient of correlation of an AR(1) random field in an one-dimensional space is given by

\[
\rho(t_1, t_2) = \rho(\tau) = (1 + \frac{\tau}{L})e^{-\frac{\tau}{L}}
\]

and in a two-dimensional space is given by

\[
\rho(t_1, t_2) = \rho(\tau) = \frac{\tau}{L}K_0 \left(-\frac{\tau}{L}\right)
\]

where \( \tau = ||t_2 - t_1|| \), \( L \) is the correlation length and \( K_0 \) is the modified Bessel function of the second kind of order 0.

To be used in the finite element method, the random field must be discretized into a finite number of random variables. The random field discretisation method used was the mid-point method.
This is one of the simplest methods, where the field value over an element is taken to be equal to the value at the mid-point of that element.

7 APPLICATIONS

The first application is a rectangular cantilever beam with length $l = 1\text{m}$, width $b = 20\text{mm}$, Young's modulus $E = 210\text{GPa}$, and mass density $\rho = 7000\text{Kg/m}^3$, as illustrated on figure 4. The thickness is uncertain and is modelled as a first-order auto-regressive AR(1) random field with mean $\mu_h = 2\text{mm}$, coefficient of variation (COV) $h_E = \sigma_h/\mu_h = 0.05$ and correlation length $L = 0.1\text{mm}$. The beam was discretized in 20 elements, except where stated otherwise.

The second application is a simply supported rectangular plate with dimensions $1 \times 0.8\text{m}^2$ and the same material properties as the beam application. The thickness is again uncertain and is modelled as a first-order auto-regressive AR(1) random field with mean $\mu_h = 2\text{mm}$, coefficient of variation (COV) $h_E = \sigma_h/\mu_h = 0.05$ and correlation length $L = 0.1\text{mm}$.

The comparison of the natural frequencies standard deviation obtained by the different methods is shown in figures 5 and 6. The results obtained by the Monte Carlo and perturbation methods are quite similar while the standard deviations obtained with the fuzzy method are slightly overestimated.

Figure 7 shows the influence of the coefficient of variation of the parameters. The perturbation method gives the same results as the Monte Carlo method except for higher levels of the parameters COV. The fuzzy method overestimates the results in all of the range analysed, and for an unity COV there is no result shown because the membership function of the eigenvalue crossed zero and hence has no physical meaning. Figure 8 shows the convergence of the methods as the number of parameters increase. The overestimation of the fuzzy method seems to be limited with respect of the number of parameters. Due to the scale of the figure it is possible to notice the fluctuation in the Monte Carlo results due to its statistic nature. It would be necessary more than the 100 samples used to eliminate this fluctuation.

The slow convergence of the Monte Carlo method can be seen on figure 9, but as shown in figure 10, the Monte Carlo method
becomes attractive for higher dimensional problems.

![Figure 7: Behaviour of the different methods with the increase of the parameter’s coefficient of variance (COV) of the cantilever beam](image1)

![Figure 8: Behaviour of the different methods with the increase of the number of elements/parameters of the cantilever beam](image2)

8 CONCLUSIONS

Three common uncertainty propagation methods in structural dynamics analysis have been presented and compared.

Although good results were obtained with the perturbation method, more complex structures have to be compared and using other result quantities such as FRFs in order to determine its wide applicability to structural dynamics. The fuzzy finite element method does not seem a practical solution. This approach might still be interesting in applications with a discrete number of parameters with a strong non-linear effect on the response.

The generality of the Monte Carlo method still makes it an invaluable tool to access the effects of uncertainty and validate results obtained with other methods, but it is computationally more inexpensive to use the perturbation method as an initial guess.

9 ACKNOWLEDGEMENTS

The authors acknowledge the support of the Engineering and Physical Sciences Research Council (United Kingdom) through grants GR/R34936 and GR/R26818. José Fonseca acknowledges the support of the Portuguese Foundation for Science and Technology through the scholarship SFRH/BD/7065/2001.
Figure 10: Processing time of the different methods with the number of parameters and extrapolation.

REFERENCES


